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The design storm concept in flood control design and planning

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Abstract: The design storm approach, where the subject criterion variable is evaluated by using a synthetic storm pattern composed of identical return frequencies of storm pattern input, is shown to be an effective approximation to a considerably more complex probabilistic model. The single area unit hydrograph technique is shown to be an accurate mathematical model of a highly discretized catchment with linear routing for channel flow approximation, and effective rainfalls in subareas which are linear with respect to effective rainfall output for a selected "loss" function. The use of a simple "loss" function which directly equates to the distribution of rainfall depth-duration statistics (such as a constant fraction of rainfall, or a ϕ-index model) is shown to allow the pooling of data and thereby provide a higher level of statistical significance (in estimating T-year outputs for a hydrologic criterion variable) than use of an arbitrary "loss" function. The above design storm unit hydrograph approach is shown to provide the T-year estimate of a criterion variable when using rainfall data to estimate runoff.

Key words: Unit hydrograph, effective rainfall, linear routing, link-node model, probabilistic model.

Notations

The following notation is used in this paper:

A_ω	= peak value (demand) of A for storm event ω
A_ω^δ	= demand during I_δ based on function F
α_{k_i}	= timing offsets for channel link #1 used in the linear routing technique
$\alpha_{k_i}^0$	= α_{k_i} corresponding to storm class Ω_{ω_0}
λ_{jn}^i	= effective rainfall proportion factors for subarea R_j and storm i
θ_{jn}^i	= effective rainfall timing offsets for subarea R_j and storm i
$\phi_j^i(s)$	= subarea unit hydrograph (UH) for subarea R_j and storm i
Ω	= probabilities space of all storms
P	= probability measure on Ω
Ω_{ω_0}	= specific storm class
$\eta^i(\cdot)$	= transfer function between measured effective rainfall and measured runoff, for storm i, using a Volterra integral model structure

$\eta_\omega(\cdot)$ = transfer function for annual storm ω, using F

$\eta_\omega^{\omega_0}(\cdot)$ = the restriction of η_ω to the probability space Ω_{ω_0}

$\eta_\omega(\cdot)$ = $\psi_\omega(\cdot)W_\omega$, $\psi_\omega(\cdot)$ a unit hydrograph

a_{k_l} = *proportion factors for linear routing technique, used for channel link #l*

$e_g^i(t)$ = effective rainfall measured at the rain gage site, for storm i

$eg_\omega(\cdot)$ = effective rainfall at the rain gage site for storm ω

$e_j^i(t)$ = *subarea R_j effective rainfall for storm i*

$e_\omega(\cdot)$ = synthetic storm pattern input, using F, for storm ω

$e_\omega^\delta(t)$ = $e_\omega(t)$ in I_δ; 0, otherwise

$\bar{e}_\omega^\delta(t)$ = $\bar{I}(e_\omega^\delta(\cdot))$ for t in I_δ; 0, otherwise

$\bar{e}_T^\delta$ = T-year return frequency value of $\bar{I}(e_\omega^\delta(\cdot))$

$\Delta e_\omega^\delta(t)$ = $e_\omega^\delta(t) - \bar{e}_\omega^\delta(t)$ in I_δ, 0 outside I_δ

$E_{\omega_0}^\delta(\cdot)$ = $E(\Delta e_\omega^\delta(\cdot) \mid \bar{I}(e_{\omega_0}^\delta(\cdot)))$

$\varepsilon_\omega^\delta(\cdot)$ = $\Delta e_\omega^\delta(\cdot) - E_\omega^\delta(\cdot)$

F = effective rainfall function

$\bar{I}(e_\omega^\delta(\cdot))$ = mean value of $e_\omega^\delta(\cdot)$ in I_δ

I_δ = peak storm pattern input time interval of duration δ; also, the operation of locating the peak duration δ of storm pattern input

$I(t)$ = inflow hydrograph for linear routing

M = rainfall-runoff model

$0(t)$ = outflow hydrograph for linear routing

$\bar{P}_\delta$ = mean precipitation for peak duration δ

$\bar{P}_T^\delta$ = T-year return frequency value of $\bar{P}_\delta$

$P_\omega(\cdot)$ = rainfall measured at the rain gage site, *for storm ω*

$P_i(t)$ = year i annual storm precipitation

$Q_g^i(t)$ = runoff hydrograph, for storm i, measured at the stream gage

$Q_i(t)$ = year i annual storm runoff

$Q_\omega(\cdot)$ = runoff hydrograph for storm ω

$Q_\omega^\delta(\cdot)$ = runoff hydrograph resulting from peak time interval δ of $e_\omega(\cdot)$

$Q_m^i(t)$ = m-subarea link-node model output for storm i

Q_{pT} = T-year return frequency peak flow rate

$q_j^i(t)$ = runoff hydrograph from subarea R_j, for storm i

R = total catchment

R_j = subarea in R

$S_T(t)$ = T-year design pattern input

$S_T^\delta(\cdot)$ = T-year design storm for peak time interval δ

s,t = temporal & integration variables

x_i = annual storm event for year i

$<k>$ = vector notation for subscript sequence, k

1 Introduction

In this paper, the unit hydrograph method (UH) is used to develop estimates of runoff criterion variables in the frequently occurring cases where the uncertainty in the rainfall distribution over the catchment dominates all other sources of modelling uncertainty. Indeed, the uncertainty in the precipitation distribution appears to be a limiting factor in the successful development, calibration, and application of all surface runoff hydrologic models (e.g., Loague and Freeze 1985; Beard and Chang 1979; Schilling and Fuchs 1986; Garen and Burges 1981; Nash and Sutcliffe 1970; Troutman 1982).

Schilling and Fuchs (1986) write "that the spatial resolution of rain data input is of paramount importance to the accuracy of the simulated hydrograph" due to "the high spatial variability of storms" and "the amplification of rainfall sampling errors by the nonlinear transformation" of rainfall into runoff. They recommend that a model should employ a simplified surface flow model if there are many subbasins; a simple runoff coefficient loss rate; and a diffusion (zero inertia) or storage channel routing technique.

In their study, Schilling and Fuchs (1986) reduced the rainfall data set resolution from a grid of 81 gages to 9 gages and to a single catchment-centered gage in a 1800 acre catchment. They noted that variations in runoff volumes and peak flows "is well above 100 percent over the entire range of storms implying that the spatial resolution of rainfall has a dominant influence on the reliability of computed runoff." It is also noted that "errors in the rainfall input are amplified by the rainfall-runoff transformation" so that "a rainfall depth error of 30 percent results in a volume error of 60 percent and peak flow error of 80 percent." They also write that "it is inappropriate to use a sophisticated runoff model to achieve a desired level of modeling accuracy if the spatial resolution of rain input is low". Similarly, Beard and Chang (1979) write that in their study of 14 urban catchments, complex models such as continuous simulation typically have 20 to 40 parameters and functions that must be derived from recorded rainfall-runoff data. "Inasmuch as rainfall data are for scattered point locations and storm rainfall is highly variable in time and space, available data are generally inadequate... for reliably calibrating the various interrelated functions of these complex models."

In the extensive study by Loague and Freeze (1985), three event-based rainfall-runoff models (a regression model, a unit hydrograph model, and a kinematic wave quasi-physically based (QPB) model were used on three data sets of 269 events from three small upland catchments. The three catchments were 25 acres, 2.8 square miles, and 35 acres in size, and were extensively monitored with rain gage, stream gage, neutron probe, and soil parameter site testing. For example, the 25 acre site contained 35 neutron probe access sites, 26 soil parameter sites (all equally spaced), an on-site rain gage and a stream gage. The QPB model utilized 22 overland flow planes and four channel segments. In comparative tests between the three modeling approaches to measured rainfall-runoff data it was concluded that all models performed poorly and that the QPB performance was only slightly improved by calibration of its most sensitive parameter, hydraulic conductivity. They write that the "conclusion one is forced to draw....is that the QPB model does not represent reality very well; in other words, there is considerable model error present. We suspect this is the case with most, if not all conceptual models currently in use." Additionally, "the fact that simpler, less data intensive models provided as good or better predictions than a QPB is food for thought."

Troutman (1982) also discusses the often cited difficulties with the error in precipitation measurements "due to the spatial variability of precipitation." This source of error can result in "serious errors in runoff prediction and large biases in parameter estimates by calibration of the model."

Based on the literature, the main difficulty in the use, calibration, and development, of complex models appears to be the lack of precise rainfall data and the high model sensitivity to (and magnification of) measurement errors. Nash and Sutcliffe (1970) write that "as there is little point in applying exact laws to approximate boundary conditions, this and the limited ranges of the variables encountered, suggest the use of simplified empirical relations."

By coupling to the precipitation variation the variation in loss rates due to nonhomogeneity and other factors, the resulting variation in effective rainfall rates versus distance over the catchment leads to the conclusion that the effective rainfall distribution over the catchment is a random process with respect to the available rain gage data.

In this paper, the rainfall-runoff modeling concepts are analyzed as a probabilistic problem in order to address the statistical correlation of measured rainfall and runoff. The problem setting is to develop T-year return frequency estimates of a surface runoff hydrologic criterion variable (such as peak flow rate, peak water depth behind a detention basin, etc.) at a stream gage site given the stream gage data, and also given the precipitation data from a single rain gage. It is shown that a multi-linear single area unit hydrograph (UH) method coupled with a design storm composed of identical return frequency storm pattern inputs, properly provides the distribution of peak demand of the subject hydrologic criterion variable. A focal point of this paper is to provide a detailed example problem, which not only demonstrates the generalizations developed herein, but also provides a quasi-analytic solution.

2 Catchment and data description

Let R be a free draining catchment with negligible detention effects. R is discretized into m subareas, R_j, each draining to a nodal point which is drained by a channel system. The m-subarea link-node model resulting by combining the subarea runoffs for storm i, adding runoff hydrographs at nodal points, and routing through the channel system, is denoted as $Q_m^i(t)$ where t is storm time. It is assumed that there is only a single rain gage and stream gage available for data analysis. To simplify our analysis, it is assumed that the rain gage site is monitored for a 'true' effective rainfall distribution, $e_g^i(t)$. The motivation in using a measured $e_g^i(t)$ at the rain gage site is to avoid the necessity of using a multiparameter submodel to approximate $e_g^i(t)$; rather we assume that an accurate value of $e_g^i(t)$ is available, even though this data is measured at the rain gage site which may be located outside of the catchment, and these data only apply at the rain gage site. In latter sections, the use of $e_g^i(t)$ will be generalized to the use of an arbitrary effective rainfall pattern, $e_\omega(\cdot)$, for storm ω. The stream gage data represents the entire catchment, R, and is denoted by $Q_g^i(t)$ for storm event i. Baseflow is assumed to be negligible (or a known function of time, t).

2.1 Linear effective rainfalls for subareas

The effective rainfall distribution (rainfall less losses) in R_j is given by $e_j^i(t)$ for storm i where $e_j^i(t)$ is approximated by proportions of translates of $e_g^i(t)$ by:

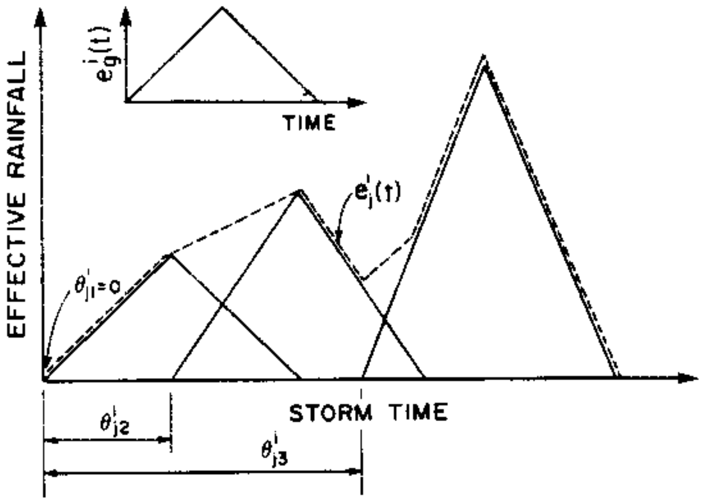


Figure 1. Subarea effective rainfall (precipitation less losses) as a linear combination of gage measured effective rainfall, $e_g^i(\cdot)$

$$e_j^i(t) = \sum_n \lambda_{jn}^i e_g^i(t-\theta_{jn}^i), \qquad j = 1,2,\ldots,m \tag{1}$$

where λ_{jn}^i and θ_{jn}^i are approximation coefficients and timing offsets, respectively, for storm i and subarea R_j. In Eq. (1), the λ_{jn}^i and θ_{jn}^i are samples of random variables for storm event i. Figure 1 illustrates the linear effective rainfall corresponding to arbitrary subarea, R_j.

2.2 *Subarea runoff*

The storm i subarea runoff from R_j, $q_j^i(t)$, is given by linear convolution:

$$q_j^i(t) = \int_{s=0}^{t} e_j^i(t-s)\phi_j^i(s)\, ds \tag{2}$$

where $\phi_j^i(s)$ is the subarea unit hydrograph (UH) for storm i such that Eq. (2) applies. Combining Eqs. (1) and (2) gives

$$q_j^i(t) = \int_{s=0}^{t} \sum_n e_g^i(t-\theta_{jn}^i-s)\lambda_{jn}^i\phi_j^i(s)\, ds. \tag{3}$$

Rearranging variables,

$$q_j^i(t) = \int_{s=0}^{t} e_g^i(t-s)\sum_n \lambda_{jn}^i\phi_j^i(s-\theta_{jn}^i)\, ds \tag{4}$$

where throughout this paper, the functions $f(t)$ which appear, and are defined for $t \geq 0$, are extended to be defined for all t by $f(t) = 0$ for $t < 0$.

2.3 *Linear routing*

Let $I_1(t)$ be the inflow hydrograph to a channel flow routing link (number 1), and $0_1(t)$ the outflow hydrograph. A linear routing model of the unsteady flow routing process is given by (e.g., see Doyle et al. 1983),

$$0_1(t) = \sum_{k_1=1}^{n_1} a_{k_1} I_1(t - \alpha_{k_1}) \tag{5}$$

where the a_{k_1} are coefficients with sum to unity; the α_{k_1} are timing offsets; and n_1 and k_1 refer to the summation index for link number 1. Again, $I_1(t - \alpha_{k_1}) = 0$ for $t < \alpha_{k_1}$. Given gage data for $I_1(t)$ and $0_1(t)$, the best fit values for the a_{k_1} and α_{k_1} can be computed by least squares optimization.

Should the above outflow hydrograph, $0_1(t)$, now be routed through another link (number 2), then $I_2(t) = 0_1(t)$ and from the notation used previously,

$$0_2(t) = \sum_{k_2=1}^{n_2} a_{k_2} I_2(t - \alpha_{k_2}) = \sum_{k_2=1}^{n_2} a_{k_2} \sum_{k_1=1}^{n_1} a_{k_1} I_1(t - \alpha_{k_1} - \alpha_{k_2}). \tag{6}$$

For L link, each with their own respective gage data, the above linear routing results in the outflow hydrograph, $0_L(t)$, being given by

$$0_L(t) = \sum_{k_L=1}^{n_L} a_{k_L} \sum_{k_{L-1}=1}^{n_{L-1}} a_{k_{L-1}} \cdots \sum_{k_2=1}^{n_2} a_{k_2} \sum_{k_1=1}^{n_1} a_{k_1} I_1(t - \alpha_{k_1} - \alpha_{k_2} - \cdots - \alpha_{k_{L-1}} - \alpha_{k_L}). \tag{7}$$

Using a different notation, the above $0_L(t)$ is written as

$$0_L(t) = \sum_{<k>} a_{<k>} I_1(t - \alpha_{<k>}), \tag{8}$$

where $<k>$ implies the repeated summations shown in Eq. (7).

For subarea R_j, the runoff hydrograph for storm i, $q_j^i(t)$, flow through L_j links before arriving at the stream gage and contributing to the total modeled runoff hydrograph, $Q_m^i(t)$. All of the constant $a_{<k>}^i$ and $\alpha_{<k>}^i$ are available on a storm by storm basis. Consequently from the linearity of the routing technique, the m-subarea link-node model is given by the sum of the m, $q_j^i(t)$ contributions,

$$Q_m^i(t) = \sum_{j=1}^{m} \sum_{<k>_j} a_{<k>_j}^i q_j^i(t - \alpha_{<k>_j}^i) \tag{9}$$

where each vector $<k>_j$ is associated to a R_j, and all data is defined for storm i. It is noted that in all cases, continuity of volume is preserved by

$$\sum_{<k>_j} a_{<k>_j}^i = 1. \tag{10}$$

The linear routing technique of Eq. (5) is a variant of the stream flow routing convolution technique of Doyle et al. (1983). As noted in Doyle et al. (1983), different sets of calibration parameters would be needed for different classes of hydrographs (e.g., low-flow hydrographs versus high-flows). However for specified ranges or classes of hydrographs, a single set of routing parameters may be appropriate. Hence, on a hydrograph class basis, the routing effects are essentially linear and are adequately described by the model of Eq. (8).

The assumptions involved in the above derivations (i.e., that the routing effects are approximately linear for classes of hydrographs, and that a single set of calibrated routing parameters are appropriate for a class of hydrographs) will be useful in the latter sections of this paper when developing estimates of hydrologic criterion variables and their probabilistic distributions.

2.4 Link-node model $Q_m^i(t)$

For the above linear approximations for storm i, Eqs. (1), (4), and (9) can be combined to give the final form for the m subarea link-node model, $Q_m^i(t)$,

$$Q_m^i(t) = \sum_{j=1}^{m} \sum_{<k>_j} a^i_{<k>_j} \int_{s=0}^{t} e_g^i(t-s) \sum_n \lambda^i_{jn} \phi^i_j(s - \theta^i_{jn} - \alpha^i_{<k>_j})\, ds. \tag{11}$$

Because the measured effective rainfall distribution, $e_g^i(t)$, is independent of the several indices, Eq. (11) is rewritten in the form

$$Q_m^i(t) = \int_{s=0}^{t} e_g^i(t-s) \sum_{j=1}^{m} \sum_{<k>_j} a^i_{<k>_j} \sum_n \lambda^i_{jn} \phi^i_j(s - \theta^i_{jn} - \alpha^i_{<k>_j})\, ds \tag{12}$$

where all parameters are evaluated on a storm by storm basis, i.

Equation (12) describes a model which represents the total catchment runoff response based on variable subarea UH's, $\phi^i_j(s)$; variable effective rainfall distributions on a subarea-by-subarea basis with differences in magnitude (λ^i_{jn}); timing, (θ^i_{jn}), such as due to offsets in timing between the measured data and the subarea data, and pattern shape (linearity assumption); and channel flow routing translation and storage effects (parameters $a^i_{<k>_j}$ and $\alpha^i_{<k>_j}$). All parameters employed in Eq. (12) must be evaluated by runoff data where stream gages are supplied to measure runoff from each subarea, R_j, and stream gages are located upstream and downstream of each channel reach (link) used in the model.

2.5 Model reduction

The m-subarea model of Eq. (12) is directly reduced to the simple single area UH model (no discretization of R into subareas) given by $Q_1^i(t)$ where

$$Q_1^i(t) = \int_{s=0}^{t} e_g^i(t-s) \eta^i(s)\, ds \tag{13}$$

where $\eta^i(s)$ is a realization of a stochastic process for storm event i.

From Eq. (13) it is seen that the classic single area UH model is equivalent to to the highly complex link-node modeling structure of Eq. (12), where considerable runoff gage data is supplied interior of the catchment, R, so that all modeling parameters are accurately calibrated on a storm-by-storm basis. For the case of having available only a single rain gage site (where the effective rainfall is measured, $e_g^i(t)$) and a stream gage for data correlation purposes, the $\eta^i(s)$ properly represents the several effects used in the development leading to Eq. (12). Because the $Q_1^i(t)$ model structure actually includes most of the effects which are important in flood control hydrologic response, it can be used to develop useful probabilistic distributions of hydrologic modeling output.

In comparing the two models, (12) and (13), it is noted that $Q_m^i(t) = Q_1^i(t)$ only when interior runoff data is supplied to accurately evaluate all the modeling parameters used in Eq. (12). Should the catchment be discretized into many small subareas with small channel routing links (e.g., such as used in highly subdivided catchments with UH approximations, or as employed in kinematic wave (KW) type models such as MITCAT, or the KW version of HEC-1 (see HEC TD #15 1982)), then with a stream gage located at each subarea (or overland flowplane) and at each channel link, all modeling parameters could be accurately evaluated on a storm-by-storm basis, resulting in the formulation of Eq. (12).

Indeed, only by means of subarea stream gage data can the subarea linear effective rainfall distribution parameters of λ_{jn}^{i} and θ_{jn}^{i} be accurately determined for each storm event i. But it is these linear effective rainfall distribution parameters that reflect the important spatial and temporal variability of storm rainfall over the catchment which in turn causes the major difficulties in the development, calibration, and use, of hydrologic models (Schilling and Fuchs 1986: Loague and Freeze 1985; among others).

It is assumed in this paper that only a single rain gage (which is monitored to accurately develop the effective rainfall at the rain gage site, $e_g^i(t)$) and stream gage are available for data analysis. Consequently, any hydrologic model serves to correlate the data pair $\{e_g^i(t), Q_g^i(t)\}$ for each storm event i.

By approximating the parameters, the estimator model, $\hat{Q}_m^i(t)$, cannot achieve the accuracy of $Q_1^i(t)$, and $\hat{Q}_m^i(t) \neq Q_1^i(t)$.

From the above, the simple single area UH model, $Q_1^i(t)$, properly represents the appropriate UH for each subarea (or overland flow plane) for storm i; the appropriate linear routing parameters for each channel link, for storm i; the appropriate timing offsets and proportions of the measured effective rainfalls, for each subarea; and the appropriate summation of runoff hydrographs at each confluence. In contrast, the model estimator, $\hat{Q}_m^i(t)$, uses estimates for all of the parameters and subarea effective rainfall factors, and hence cannot achieve the accuracy of $Q_1^i(t)$ without the addition of interior runoff data to accurately validate the parameter values.

3 Storm classification system

Consider a catchment which has one rain gage and one stream gage. The underlying probability space Ω is the space of all rain storms ω occurring over the catchment area. A storm is a very complex event, having a beginning time, duration and variation in spatial extent and spatial intensity. Historical records do not give detailed information about the storm ω: those records which are available usually consist of (1) the record of precipitation $P_\omega(\cdot)$ from the rain gage, whose value at time t is $P_\omega(t)$, which is a realization of the stochastic process $\{P_\omega(t){:}t \text{ real}\}$, and (2) the record of discharge $Q_\omega(\cdot)$ at the stream gage, whose value at time t is, again $Q_\omega(t)$, and which is a realization of the stochastic process $\{Q_\omega(t){:}t \text{ real}\}$. The stochastic processes $P_\omega(\cdot)$ and $Q_\omega(\cdot)$ can generally be considered to be nonnegative piecewise continuous functions which vanish outside a finite interval of time.

From $P_\omega(\cdot)$ the effective rainfall at the gage, $eg_\omega(\cdot)$, is derived as will be discussed later. Applying the unit hydrograph model gives

$$Q_\omega(t) = \int_0^t eg_\omega(t-s)\eta_\omega(s)\,ds. \tag{14}$$

By a storm class Ω_{ω_0} is meant the collection at all storms with same effective rainfall as storm ω_0:

$$\Omega_{\omega_0} = \{\omega{:}eg_\omega(\cdot) = eg_{\omega_0}(\cdot)\}. \tag{15}$$

In practice one could consider those storms ω with, say $|eg_\omega(t) - eg_{\omega_0}(t)|$ arbitrarily small for all t. Given this storm class Ω_{ω_0}, we may consider it as a probability space with the induced probability measure $P_0(A) = P(A \cap \Omega_{\omega_0})/P(\Omega_{\omega_0})$

(supposing $P(\Omega_{\omega_0}) \neq 0$). Let $\eta_\omega^{\omega_0}(\cdot)$ denote the restriction of $\eta_\omega(\cdot)$ to Ω_{ω_0} with this probability (i.e., the subset of events which are considered sufficiently similar to ω_0).

Since different storms can have nearly the same effective rainfall measured at the gage but still have different discharge, the function $\eta_\omega^{\omega_0}(\cdot)$ can vary with ω, i.e., it generally is a non-constant random variable.

Consequently, in the predictive mode, where one is given an assumed or design effective rainfall distribution eg_{ω_D} (which is not measured) as input, the predicted discharge is not a single runoff hydrograph, but the stochastic process

$$Q_\omega^D(t) = \int_0^t eg_{\omega_D}(t-s)\eta_\omega^{\omega_D}(s)\,ds. \tag{16}$$

Generally there is not enough data to determine the distribution of $\eta_\omega(\cdot)$ with respect to a given storm class Ω_{ω_0}, i.e., to determine $\eta_\omega^{\omega_0}(\cdot)$. Hence additional assumptions must be used. For example, one may lump more storms in a single storm class consisting of those storms ω with "similar" $eg_\omega(\cdot)$ curves; or one may transfer $\eta_\omega^{\omega_0}(\cdot)$ distributions from another rainfall runoff set, implicitly assuming that the stochastic processes $\eta_\omega(\cdot)$ are nearly identical for the two catchments. A common occurrence is the case of predicting the runoff response from a hypothetical (or design) storm effective rainfall distribution, $eg^D(\cdot)$, which is not an element of any observed storm class. In this case, another storm class distribution must be used, which implicitly assumes that the two sets of conditional η distributions are nearly identical. Consequently for a severe design storm condition, it would be preferable to develop correlation distributions using the severe historic storms which have rainfall-runoff data available for the appropriate condition of the catchment. The example problem demonstrates the above concepts.

4 Effective rainfall uncertainty and the distributions, $\eta_\omega^{\omega_0}$ and the distribution of criterion variables

From Eq. (13), the realization for storm event i, $\eta^i(s)$, includes all the uncertainty in the effective rainfall distribution over R, as well as the uncertainty in the runoff and flow routing processes. That is, $\eta^i(s)$ is a realization of the stochastic process, $\eta_\omega^{\omega_0}$, where

$$\eta^i(s) = \sum_{j=1}^{m}\sum_{<k>} a^i_{<k>_j}\sum_n \lambda^i_{jn}\phi^i_j(s-\theta^i_{jn}-\alpha^i_{<k>_j}) \tag{17}$$

and Eq. (17) applies to storm event i, and is an element of some storm class Ω_{ω_0}. For severe storms of flood control interest, one would be dealing with only a subset of the set of all storm classes. In a particular storm class, Ω_{ω_0}, it is assumed that the subarea runoff parameters and channel flow routing uncertainties are minor in comparison to the uncertainties in the effective rainfall distribution over R (e.g., Schilling and Fuchs 1986; among others).

For a highly discretized catchment model, the use of a mean value UH for each subarea, $\phi_j(s)$, has a minor influence in the total model results (Schilling and Fuchs 1986). Although use of Eq. (13) in deriving the $[\eta(s)]_0$ distributions combines the uncertainties of both the effective rainfalls and also the channel routing and other processes, equation (17) is useful in motivating the use of the probabilistic distribution concept in design and planning studies for all hydrologic models, based on just the magnitude of the uncertainties in the effective rainfall distribution over R. That is, although one may argue that a particular model is "physically based" and represents the "true" hydraulic response distributed throughout the catchment, the uncertainty in rainfall still remains and is not reduced by increasing hydraulic routing modeling complexity. Rather, the uncertainty in rainfall is reduced only by the use of additional rainfall-runoff data. In Eq. (17), the use of mean value parameters for the routing effects implicitly assumes that the variations in storm parameters of $[\lambda_{jk}]$ and $[\theta_{jk}]$ are not so large such as to develop runoff hydrographs which cannot be modeled by a single set of linear routing parameters on a channel link-by-link basis.

If the rainfall-runoff model is to be used for storms which are considered to have different linear routing parameters and subarea UH's, then it is appropriate to define routing storm classes, Ω_{R_k}, composed of all storm classes Ω_{ω_0} measured at the rain gage, which have associated restricted distributions, composed of identical linear routing parameters (on a link basis), for each storm in Ω_{R_k}. Such a classification may be based upon effective rainfalls measured at the rain gage which are considered "severe," "heavy," "mild," and "minor." Furthermore, storm events may be subdivided into the sum of storms where different $\eta(\cdot)$'s for different routing classes apply (Doyle et al. 1983).

The goal is to develop estimates of rare occurrence values of a runoff criterion variable (or operator), A, evaluated at the stream gage site. Examples of A are the peak flow rate, or a detention basin peak volume for a given outlet structure located at the stream gage.

For simplicity, let all the effects of one year's precipitation be identified with an annual storm event ω; the underlying probability space is then the space of all such annual storms. Event ω may have a duration of a few hours or a few weeks in order to include all the precipitation, $P_\omega(\cdot)$, assumed to be of importance in correlating the event ω to the stream gage measured runoff, $Q_\omega(t)$. Hence in the following development, the operation A is assumed responsive to storm events of relatively short duration (i.e., storms of duration of a few weeks or less). The vast majority of criterion variables of interest in flood control design and planning of small urban catchments are responsive to such short duration storm events.

The criterion variable of interest is noted by A_ω for event ω where

$$A_\omega = \mathrm{A}(Q_\omega(\cdot)) \tag{18}$$

For example, peak discharge is $A_\omega = \max\,\{Q_\omega(t){:}t \text{ real}\}$ and volume of discharge is $A_\omega = \int Q_\omega(t)\,dt$.

The distribution of A_ω can be estimated from a finite sample $A_{\omega_1}, A_{\omega_2}, A_{\omega_m}$, and this empirical distribution can be used to obtain the desired T-year return frequency estimates, A_T, of the criterion variable where by definition

$$P(A_\omega \geq A_T) = 1/T, \qquad \text{for } T > 1. \tag{19}$$

5 The rainfall-runoff model

With only a single rain gage available, all rainfall-runoff models must operate on the annual precipitation events $P_\omega(\cdot)$. The notation of "effective rainfall" will be generated in the following.

Let F be a function on the precipitation measured at the rain gage:

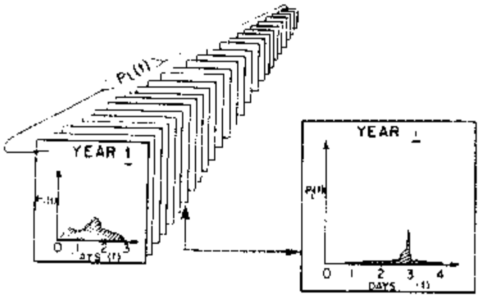


Figure 2. The sequence of annual precipitation events

$$F : P_\omega(\cdot) \rightarrow e_\omega(\cdot), \tag{20}$$

such that $e_\omega(\cdot)$ is a nonnegative, bounded, piecewise continuous function of time t. For example,

$$F : P_\omega(t) \rightarrow P_\omega^2(t); \quad F : P_\omega(t) \rightarrow \int_{s=0}^{t} P_\omega(s)\, ds. \tag{21}$$

Figures 2-4 illustrate the use of various formulations for F.

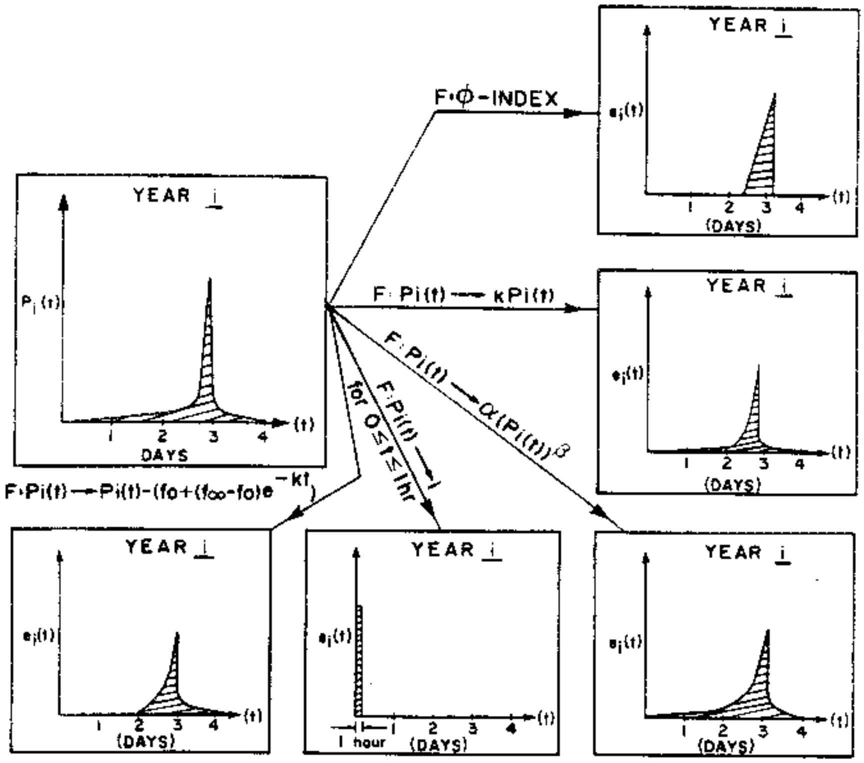


Figure 3. Example loss functions

F :

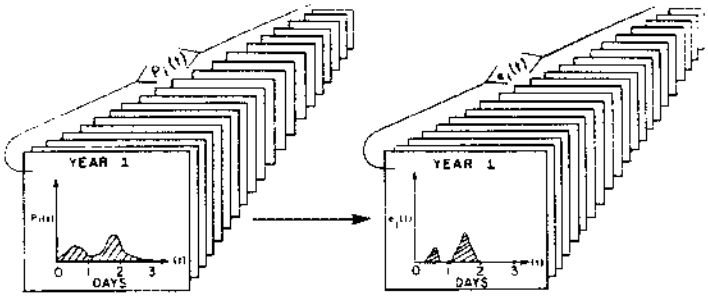


Figure 4. Sequence of annual effective rainfalls generated from the sequence of annual precipitation

The rainfall-runoff model, M is used to correlate the synthetic "effective rainfall" $e_\omega(\cdot)$ to the measured runoff, $Q_\omega(\cdot)$. Note that $e_\omega(\cdot)$ depends very strongly on the mapping F chosen.

In the single area UH model formulation of Eq. (13), and with $e_\omega(\cdot)$ in place of the term $eg_\omega(\cdot)$, the realization $\eta_\omega(s)$ now depends on both the function F and the storm event ω.

Thus for the model, M, the storm pattern input, $e_\omega(\cdot)$, and the realization, $\eta_\omega(\cdot)$, are used to equate with $Q_\omega(t)$ by

$$M : \langle e_\omega(\cdot), \eta_\omega(\cdot)\rangle \rightarrow Q_\omega(\cdot) \tag{22}$$

where $e_\omega(\cdot)$ must not be strictly zero for any ω where $Q_\omega(\cdot)$ is not strictly zero. The theorem of Titchmarch provides that if $\int_{s=0}^{t} e_\omega(t-s)\eta_\omega(s)\,ds = Q_\omega(t)$, then such an $\eta_\omega(\cdot)$ is unique assuming that $e_\omega(\cdot)$ initiates at storm time $t = 0$ (see Mikusinski 1983). From this it can be shown that given $Q_\omega(\cdot)$ and $e_\omega(\cdot)$, there is an $\eta_\omega(\cdot)$ for which Eq. (22) holds to within an arbitrarily given degree of accuracy.

5.1 Critical duration analysis

Consider a storm pattern input, $e_\omega(\cdot)$, and let I_δ be the operation of locating the δ-time interval of peak area in $e_\omega(\cdot)$. Then (see Fig. 5)

$$I_\delta : e_\omega(\cdot) \rightarrow e_\omega^\delta(\cdot) \tag{23}$$

where $e_\omega^\delta(t) \equiv 0$ for all $t \notin I_\delta$; $e_\omega^\delta(t) = e_\omega(t)$ for $t \in I_\delta$; and where $\delta > 0$. It is noted that I_δ is also used as the notation for the peak interval.

Other notation follows immediately when using $e_\omega^\delta(\cdot)$ to produce runoff (Fig. 6)

$$Q_\omega^\delta(t) = \int_{s=0}^{t} e_\omega^\delta(t-s)\eta_\omega(s)\,ds \tag{24}$$

$$A_\omega^\delta = \mathrm{A}(Q_\omega^\delta(\cdot)). \tag{25}$$

Because A_ω^δ is the peak demand from $Q_\omega^\delta(\cdot)$, A_ω^δ is a function of δ for a given storm event, ω, and also depends on the choice of F. (It is easily shown that A_ω^δ is not necessarily a nondecreasing function of δ.)

The critical duration, $D_\omega(A)$, is that value of δ such that

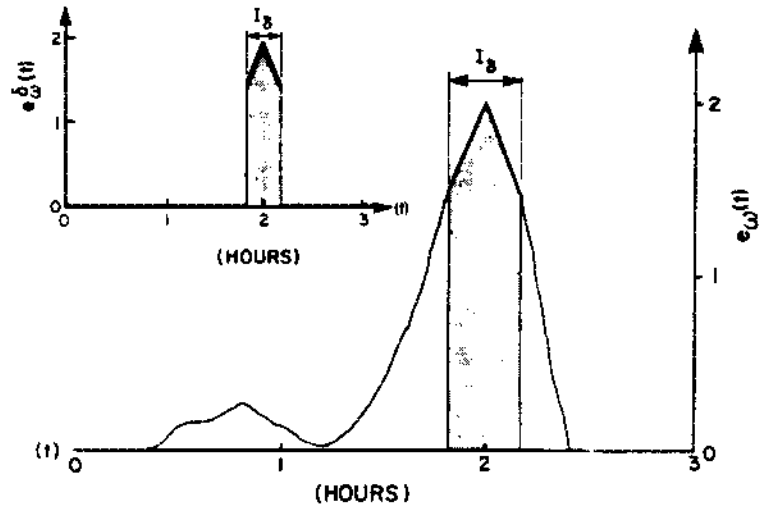


Figure 5. The function $e_\omega^\delta(\cdot)$ developed from $e_\omega(\cdot)$ for peak duration I_δ

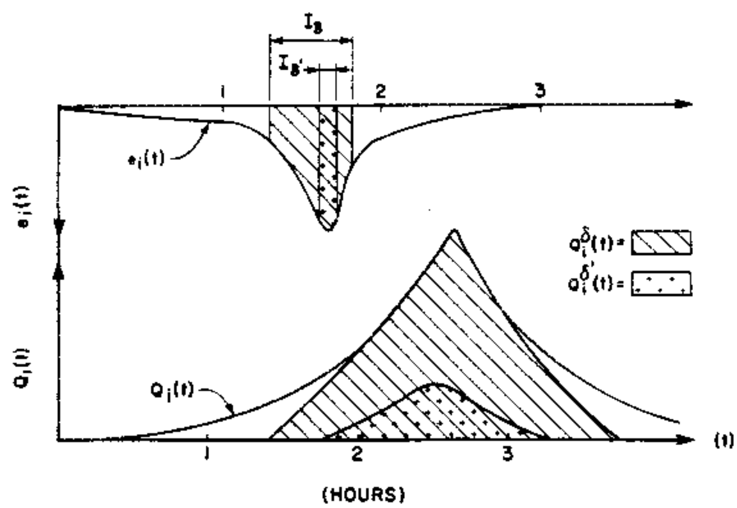


Figure 6. Two runoffs developed from two peak effective rainfalls $e_i^\delta(\cdot)$ and $e_i^{\delta'}(\cdot)$

$$D_\omega(A) = \inf\{\delta : A_\omega^\delta = A_\omega\}. \tag{26}$$

That is, for storm event ω, the critical duration is the minimum peak time duration (of size δ) such that the model estimate of the runoff criterion variable is maximum.

5.2 *Criterion variable distribution analysis*

From the above

$$A_\omega = \mathrm{A}\left(\int_{s=0}^{t} e_\omega(t-s)\eta_\omega(s)\, ds\right) \tag{27}$$

and

$$A_{\omega}^{\delta} = \mathrm{A}(\int_{s=0}^{t} e_{\omega}^{\delta}(t-s)\eta_{\omega}(s)\, ds) \tag{28}$$

where $e_{\omega}^{\delta}(\cdot) \to e_{\omega}(\cdot)$ as $\delta\uparrow$ (i.e., as δ increases from zero). Then $A_{\omega}^{\delta} \to A_{\omega}$ as $\delta\uparrow$ where reasonable assumptions of continuity on A are assumed.

The $Q_{\omega}^{\delta}(\cdot)$ hydrograph for peak duration I_{δ} and function F is given by

$$Q_{\omega}^{\delta}(t) = \int_{s=0}^{t} e_{\omega}^{\delta}(t-s)\eta_{\omega}(s)\, ds \tag{29}$$

where $\eta_{\omega}(\cdot)$ is that η which correlates the total model input $e_{\omega}(\cdot)$ with the discharge $Q_{\omega}(\cdot)$. Note there may be several runoff hydrographs to be correlated to a single storm pattern input, $e_{\omega}(\cdot)$, and hence even if $e_{\omega}(\cdot) = e_{\omega'}(\cdot)$, $\eta_{\omega}(\cdot)$ may not be equal to $\eta_{\omega'}(\cdot)$.

It is useful to rewrite $\eta_{\omega}(\cdot)$ into the standard unit hydrograph formulation,

$$\eta_{\omega}(\cdot) = W_{\omega}\psi_{\omega}(\cdot) \tag{30}$$

where $\psi_{\omega}(\cdot)$ is the unit hydrograph (UH) associated to a particular storm pattern input which depends on F; and W_{ω} is the ratio of total storm runoff to total storm pattern input mass, namely, $W_{\omega} = \int_{s=0}^{\infty} Q_{\omega}(s)\, ds / \int_{s=0}^{\infty} e_{\omega}(s)\, ds$. Thus for the distributions $\eta_{\omega}(\cdot)$, distribution W_{ω} and $\psi_{\omega}(\cdot)$ are associated by means of Eq. (30).

Define components $\bar{e}_{\omega}^{\delta}(\cdot)$ and $\Delta e_{\omega}^{\delta}(\cdot)$ by (see Figs. 7 and 8),

$$\bar{I}(e_{\omega}^{\delta}(\cdot)) = \frac{1}{\delta}\int_{s=0}^{\infty} e_{\omega}^{\delta}(s)\, ds = \frac{1}{\delta}\int_{I_{\delta}} e_{\omega}(s)\, ds \tag{31}$$

$$\bar{e}_{\omega}^{\delta}(t) = \begin{cases} \bar{I}(e_{\omega}^{\delta}(\cdot)), & \text{for } t \in I_{\delta} \\ 0, & \text{otherwise} \end{cases} \tag{32}$$

$$\Delta e_{\omega}^{\delta}(t) = \begin{cases} e_{\omega}^{\delta}(t) - \bar{e}_{\omega}^{\delta}(t), & t \in I_{\delta} \\ 0, & \text{otherwise} \end{cases} \tag{33}$$

From Eq. (31), the random variable $\bar{I}(e_{\omega}^{\delta}(\cdot))$ T-year return frequency values are denoted by $\bar{e}_{T}^{\delta}$, where $P(\bar{I}(e_{\omega}^{\delta}(\cdot)) \geq \bar{e}_{T}^{\delta}) = 1/T$ (see Fig. 8). From Eqs. (29) to (33), the distribution of runoff hydrographs associated with peak interval I_{δ} is given by

$$Q_{\omega}^{\delta}(t) = \int_{s=0}^{t} ([\bar{e}_{\omega}^{\delta}(s) + \Delta e_{\omega}^{\delta}(s)] W_{\omega}\psi_{\omega}(t-s)\, ds \tag{34}$$

where, given F, a peak duration I_{δ}, and an average input intensity $\bar{I}(e_{\omega}^{\delta}(\cdot))$, $\Delta e_{\omega}^{\delta}(\cdot)$ is the distribution of input storm patterns about $\bar{e}_{\omega}^{\delta}(\cdot)$.

For a given function F, the random variable W_{ω} and stochastic process $\psi_{\omega}(\cdot)$ include the important random variations in effective rainfall distributions and magnitudes over R, with respect to the storm pattern input, $e_{\omega}(\cdot)$. But these components are seen to be a constant fraction loss function and the standard unit hydrograph which are both well-known, frequently used concepts.

We are interested in the "average shape" of those storms which have, for a given I_{δ}, the same average effective rainfall intensity. To proceed, let $\Omega_{\omega_0}^{\delta} = \{\omega : \bar{I}(e_{\omega}^{\delta}(\cdot)) = \bar{I}(e_{\omega_0}^{\delta}(\cdot))\}$ (In practice one would look at storms which have "nearly equal" intensities.) Then, if $P(\Omega_{\omega_0}^{\delta}) \neq 0$, the "average shape" involved is

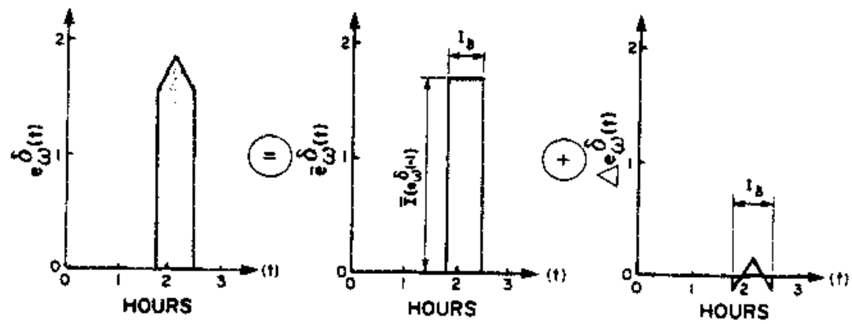


Figure 7a. Definition of $e_\omega^\delta(\cdot)$ components

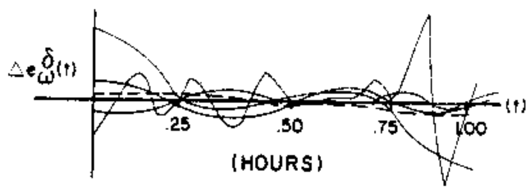


Figure 7b. Typical plots of $\Delta e_\omega^\delta(\cdot)$

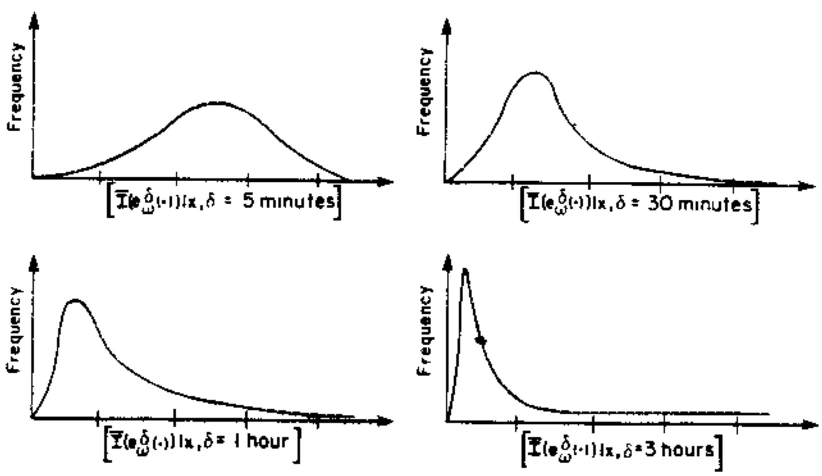


Figure 8. Distributions of $\bar{I}(e_\omega^\delta(\cdot))$ for several durations, δ

determined by considering (see example problem for demonstration)

$$\frac{1}{P(\Omega_{\omega_0}^\delta)}\int_{\Omega_{\omega_0}^\delta} \Delta e_\omega^\delta(t)P(d\omega) \tag{35}$$

with $P(d\omega)$ being the probability measure on Ω.

Note that I_δ varies with ω, so before integrating in Eq. (35), translate all the $\Delta e_\omega^\delta(\cdot)$ curves to $[0,\delta]$; this intermediate step only translates the associated $Q_\omega^\delta(\cdot)$ curves. The more general way to proceed, which applies even when $P(\Omega_{\omega_0}^\delta) = 0$, is to use the conditional expectation:

$$E(\Delta e^{\delta}_{\omega}(t) \,|\, \bar{I}(e^{\delta}_{\omega_0}(\cdot))) \tag{36}$$

which, for each t, agrees with Eq. (35) when $P(\Omega^{\delta}_{\omega_0}) \neq 0$. As an abbreviation, introduce the notation

$$E^{\delta}_{\omega_0}(t) = E(\Delta e^{\delta}_{\omega}(t) \,|\, \bar{I}(e^{\delta}_{\omega_0}(\cdot))). \tag{37}$$

Define $\varepsilon^{\delta}_{\omega}(t)$ by

$$\Delta e^{\delta}_{\omega}(t) = E^{\delta}_{\omega}(t) + \varepsilon^{\delta}_{\omega}(t). \tag{38}$$

Then

$$Q^{\delta}_{\omega}(t) = \int_0^t (\bar{e}^{\delta}_{\omega}(s) + E^{\delta}_{\omega}(s) + \varepsilon^{\delta}_{\omega}(s))\eta_{\omega}(t-s)\, ds. \tag{39}$$

5.3 The choice of mapping F

With limited data, the choice of the mapping F has a significant impact on $Q^{\delta}_{\omega}(\cdot)$ and A^{δ}_{ω}. We want $A^{\delta}_{\omega} \to A_{\omega}$ as $\delta\uparrow$, and an arbitrary choice of F can result in a poorer correlation between rainfall-runoff data then another choice for F.

In developing the distributions for $\bar{e}^{\delta}_{\omega}(\cdot)$ and $\Delta e^{\delta}_{\omega}(\cdot)$, it is often useful to supplement the available data by including additional sources of data such as obtained from other nearly identical watersheds where the rainfall-runoff model, M, has already been applied. For a general function F, however, such additional data is generally unavailable.

However, if one uses an F function definition which results in a one-to-one mapping between the $\bar{e}^{\delta}_{\omega}(\cdot)$ distributions and the corresponding depth-duration distributions for precipitation, $\bar{P}^{\delta}_{\omega}(\cdot)$, then usually a considerable advantage is afforded as the rainfall data has often been locally regionalized (see Fig. 9).

EXAMPLE: Let $F: P_{\omega}(t) \to kP_{\omega}(t)$, k a constant > 0 .
Let $\bar{P}^{\delta}_{\omega}(\cdot)$ be a δ-duration rainfall distribution. Then $\bar{e}^{\delta}_{\omega}(\cdot) = k\bar{P}_{\delta}(\cdot)$ for all $\delta > 0$.
Let $F: P_{\omega}(t) \to P_{\omega}(t) - \phi$, when positive; 0, otherwise.
Then for durations δ where $e^{\delta}_{\omega}(\cdot)$ is nonzero almost everywhere, $\bar{e}^{\delta}_{\omega}(\cdot) = \bar{P}^{\delta}_{\omega}(\cdot) - \phi$.

From the above examples, the well-known constant fraction "loss" function and the phi-index "loss" function both develop $\bar{e}^{\delta}_{\omega}(\cdot)$ which can be immediately equated to regional $\bar{P}^{\delta}_{\omega}(\cdot)$ information. Note that use of a more "physically based" F function such as, for example, the Horton equation results in the $\bar{e}^{\delta}_{\omega}(\cdot)$ being dependent upon when in the storm pattern time the peak precipitation bursts occur; hence, in this case $\bar{e}^{\delta}_{\omega}(\cdot)$ could not be directly equated to the $\bar{P}^{\delta}_{\omega}(\cdot)$.

Once $\bar{I}(e^{\delta}_{\omega}(\cdot))$ are known, the T-year exceedance values, $\bar{e}^{\delta}_{T}$, are given by $P(\bar{I}(e^{\delta}_{\omega}(\cdot)) \geq \bar{e}^{\delta}_{T}) = 1/T$.

5.4 The "design storm" input concept

In this section, the well-known design storm approach to estimating T-year values of a criterion variable (in rainfall-runoff models) is developed. Because all rainfall-runoff modeling components are included in the following probabilistic analysis, the resulting equations are generalized. Let a mapping F be given. Then $Q^{\delta}_{\omega}(t)$ is given by Eq. (39) and the criterion variable $A^{\delta}_{\omega} = \mathrm{A}(Q^{\delta}_{\omega}(\cdot)) \to A_{\omega}$ as $\delta\uparrow$.

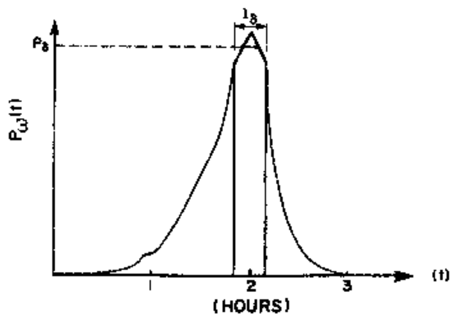


Figure 9. The mean precipitation intensity, P_δ, for duration l_δ

In the integrand in Eq. (39), the sum

$$\bar{e}_\omega^\delta(\cdot) + E_\omega^\delta(\cdot) \tag{40}$$

is the mean intensity of the storm ω, over l_δ, plus the "expected shape" among those storms having this intensity. Given a return frequency T, not necessarily an integer, assume there is some storm ω so that for this

$$\bar{I}(e_\omega^\delta(\cdot)) = \bar{e}_T^\delta$$

$\bar{e}_T^\delta$ being the T-year value for the mean intensity $\bar{I}(e_\omega^\delta(\cdot))$ (see following Eq. (36)). For any storm ω with $\bar{I}(e_\omega^\delta(\cdot)) = \bar{e}_\omega^\delta$,

$$A_\omega^\delta = \mathrm{A}\{\int_0^t (\bar{e}_\omega^\delta(s) + E_\omega^\delta(s) + \varepsilon_\omega^\delta(s))\eta_\omega(t-s)\, ds\}$$

and the T-year design storm, for δ, is defined by

$$S_T^\delta(t) = \bar{e}_T^\delta(t) + E_T^\delta(t) \tag{41}$$

for this ω, or any other ω with $\bar{I}(e_\omega^\delta(\cdot)) = \bar{e}_T^\delta$. "The" T-year design storm $S_T(\cdot)$ is thereby considered to be a collection of storms $S_T^\delta(\cdot)$, one for each $\delta > 0$.

From Eq. (41), the "design storm" concept is embodied in the formulation

$$A_\omega^\delta = \mathrm{A}[\int_{s=0}^t (S_T^\delta(s) + \varepsilon_\omega^\delta(s))\eta_\omega(t-s)\, ds] \tag{42}$$

where $A_\omega^\delta \rightarrow \mathrm{A}_\omega$ as $\delta\uparrow$, and T varies independently of δ. The following simplifications of Eq. (42) illustrate the concepts embodied in the design storm approach.

6 Design storm model simplifications

Let $R_k(\omega)$ be the routing class to which ω belongs. The range of R_k can be regarded as a finite sequence of outcomes and so a subset of E^n for some fixed value of n. It is useful to consider the probabilistic average η for each routing class, i.e., the conditional expectation $E(\eta_\omega(\cdot) \mid R_k)$.

A basic assumption is that the effects of this averaging together of $\eta_\omega(\cdot)$ with the variation in $\varepsilon_\omega^\delta(\cdot)$, combine so as to enable the simplification of the distribution of A_ω^δ (approximately) by

$$A_\omega^\delta \simeq [\int_0^t S_T^\delta(s)E(\eta_\omega(t-s) \mid R_k)\, ds]. \tag{43}$$

And as $\delta\uparrow$, this will converge in distribution to A_ω, (i.e., A_ω varies as a function of the random variable, T),

$$A_\omega \overset{D}{=} \max_\delta [\mathrm{A}\int S_T^\delta(s)E(\eta_\omega(t-s) \mid R_k)\, ds] \tag{44}$$

(i.e., it is assumed that both sides of Eq. (44) have the same probability distribution; and δ and T are independent variables).

If furthermore, the peak demand monotonically increases with respect to increasing T used in the argument $S_T^\delta(\cdot)$ in Eq. (44), then the A_{T_0} point estimates for the distribution of A_ω are obtained by using $S_{T_0}^\delta(\cdot)$ in Eq. (44).

In Eqs. (41)-(44), the distribution of A_ω is determined by evaluating the distributions of A_ω^δ for successively larger δ (as $\delta\uparrow$). As δ increases, a different $S_T^\delta(\cdot)$ is determined such as shown in Figs. 10a and 10b. Thus Eq. (44) represents an algorithmic procedure where successively larger δ values are used to determine A_ω^δ until A_ω is determined.

Equation (44) is the basis of the widely used design storm approach for estimating T-year return frequency values of the random variable A_ω (e.g., Hromadka and McCuen 1986; Beard and Chang 1979; U.S. Army Corps. of Engineers, HEC TD#15 1982; among others).

Now consider the formulation of Eq. (44) and approximate $E(\eta_\omega(\cdot) \mid R_k)$ by

$$E(\eta_\omega(\cdot) \mid R_k) = E(W_{R_k})E(\psi_\omega(\cdot) \mid R_k) \tag{45}$$

where $E(W_{R_k})$ and $E(\psi_\omega(\cdot) \mid R_k)$ follow from Eq. (30). If F is defined on a storm class basis by the constant fraction relationship, $F: P_\omega(t) \rightarrow \lambda_{R_k}P_\omega(t)$, λ_{R_k} constants > 0, then Eqs. (44) and (45) are combined as

$$A_\omega^\delta = \mathrm{A}(\int_0^t C_{R_k}P_T^\delta(s)E(\psi(t-s) \mid R_k)\, ds) \tag{46}$$

where $C_{R_k} = \lambda_{R_k}E(W_{R_k})$; $P_T(\cdot)$ is the "T-year" design storm rainfall pattern such that for each I_δ, the average intensity of $P_T^\delta(\cdot)$ equals the T-year return frequency rainfall for peak duration δ, $\bar{P}_T^\delta$, and the shape of $P_T^\delta(\cdot)$ matches the expected shape of rainfall patterns corresponding to the T-year depth for a given δ (see Figs. 10a and 10b).

Should the catchment hydraulics respond similarly over the range of storms considered significant for the criterion variable (hence partially eliminating the need for routing classes), and also the variation of effective rainfall over R be similar (i.e., W_{R_k}) over the subject range of storms, then Eq. (46) is simplified to

$$A_\omega^\delta = \mathrm{A}(C\int_{s=0}^t P_T^\delta(s)\psi(t-s)\, ds) \tag{47}$$

which converges in distribution to A_ω as $\delta\uparrow$; where C is a positive constant and $\psi(\cdot)$ is a unit hydrograph. Finally, should the time patterns of $P_T^\delta(\cdot)$ be similar over the range of storms considered (i.e., on a return frequency basis, T), and also the critical durations associated to the criterion variable be sufficiently small such that expected time patterns of $P_T^\delta(\cdot)$ are similar (for the critical durations of interest), then the criterion variable of peak flow rate, Q_{pT}, is given by a form of the well-

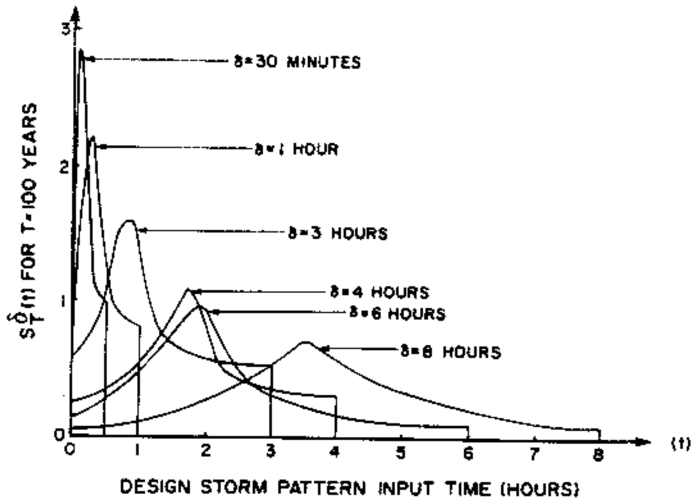


Figure 10a. Plots of $S_T^\delta(\cdot)$, for $T = 100$ years, for several durations

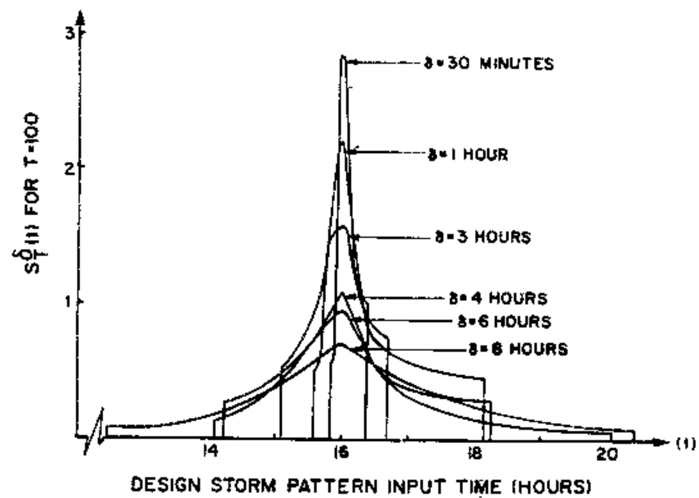


Figure 10b. Plots of $S_T^\delta(\cdot)$, for $T = 100$ years, realigned with peak intensities in alignment at storm time of 16 hours

known rational equation.

7 Limited data analysis and the "T-year" design storm

Without the entire universe of samples available for each distribution, a limited data analysis is needed in order to approximate the various distributions used to develop A_ω^δ.

For $\eta_\omega(\cdot)$, it is assumed that routing classes, R_k, apply where R_k is determined by δ, $\bar{e}_\omega^\delta(\cdot)$, and $\Delta e_\omega^\delta(\cdot)$. The distributions are then developed by determining samples, η_{ω_i}, given the function F applied to precipitation measured at the rain gage, and runoff measured at the stream gage.

The $\Delta e_{\omega}^{\delta}(\cdot)$ is evaluated using information from local studies on the storm pattern inputs such as prepared for precipitation by Huff (1967). The $\bar{e}_{\omega}^{\delta}(\cdot)$ is evaluated from depth-duration-frequency analysis of $\bar{I}(e_{\omega}^{\delta}(\cdot))$ analogous to the depth-duration-frequency analysis of precipitation, P_{ω}^{δ}. (It is recalled that $\Delta e_{\omega}^{\delta}(\cdot)$ reflects the characteristics of $F: P_{\omega}(\cdot) \rightarrow e_{\omega}(\cdot)$ applied at the rain gage.) Information regarding $E_{\omega}^{\delta}(\cdot)$ can oftentimes be inferred from the expected precipitation time patterns such as given in studies by Pierrehumbert (1974). It is noted that in both of the above references, the distributions (or the expected value) of rainfall time patterns are shown to be dependent upon the duration δ.

Should F be chosen which relates directly to the precipitation statistical characteristics (e.g., constant fraction model, $F: P_{\omega}(t) \rightarrow kP_{\omega}(t)$, $k > 0$), then the above studies on precipitation time patterns can be used directly to determine $\Delta e_{\omega}^{\delta}(\cdot)$. Additional, $\bar{e}_{\omega}^{\delta}(\cdot)$ can also be derived directly from the precipitation data. Hereafter, such a function F is said to be "conservative" with respect to rainfall.

HEC Training Document #15 (1982) provides a precipitation design storm pattern composed of identical return frequencies of rainfall nested about hour 12 of a 24-hour rainfall time pattern. Also used are depth-area adjustments on the rainfall which result in the highly peaked rainfall time pattern becoming less "peaked" as drainage area increases. This procedure can also be interpreted as a technique to develop the T-year design storm pattern, input, $S_T(t)$, when using a function F which is conservative. The depth-area adjustments modify the time pattern of $S_T(t)$ based on catchment area, whereas in the development leading to Eq. (42), $S_T(t)$ is dependent of area (for our single rain gage problem) and is a function of input peak duration, I_{δ}, and magnitude, $\bar{I}(e_{\omega}^{\delta}(\cdot))$. However, larger catchment areas typically have associated larger critical durations, and hence the depth-area adjustments provide an approximation of the effects of both $\bar{e}_{\omega}^{\delta}(\cdot)$ and $E_{\omega}^{\delta}(\cdot)$ used to define $S_T^{\delta}(\cdot)$ in Eq. (42).

Use of a conservative function F affords the advantage that both $\bar{e}_{\omega}^{\delta}(\cdot)$ and $\Delta e_{\omega}^{\delta}(\cdot)$ are directly related to precipitation data. Rainfall data is typically locally regionalized and oftentimes represents considerable station-year data. Use of a nonconservative function F results in the necessity of applying the model to the total data set used in regionalizing the rainfall data in order to achieve similar levels of statistical significance. (This is of special concern when the peak demand of the criterion variable is most sensitive to the peak durations of input (i.e., critical durations) which have limited data available.)

In the formulation of Eq. (44), it is commonplace in currently available design methods to substitute a single design storm pattern of nested "T-year" inputs (based upon T-year rainfalls, when the function F is conservative) in place of the algorithm of $A_{\omega}^{\delta} \rightarrow A_{\omega}$ as $\delta\uparrow$ (e.g., Beard and Chang 1979). However, the use of depth-area curves (or equivalent) such as applied in the HEC TD-15 may be necessary in order to approximate the variation in $S_T^{\delta}(\cdot)$ (due to $E_{\omega}^{\delta}(\cdot)$ variations) as $\delta\uparrow$.

8 Computational problem: the design storm approach

An example problem will be considered which represents, in a broad way, real hydrologic phenomena yet is simple enough to allow exact calculations of many of the hydrological and statistical parameters introduced above. It is emphasized that this example problem was developed to provide a plausible hydrologic rainfall-runoff correlation problem with a known solution, yet attempts to be realistic as to generally known (or assumed) data trends.

Suppose that the function F, as discussed above, operates on any given precipitation event, defined for $t \geq 0$, by developing an effective rainfall of the form

$$e_\omega(t) = c(e^{-at} - e^{-bt}) \qquad b > a > 0, \quad c > 0. \tag{48}$$

(Equation (48) can be considered as the result of least squares fitting to some effective rainfall.) The dependence of a, b and c on the storm ω will be understood without the more explicit notation a_o, b_o, c_o.

For the storm ω, $Q_\omega(\cdot)$ is related to $e_\omega(\cdot)$ by convolution with $\eta_\omega(\cdot)$:

$$Q_\omega(t) = \int_0^t e_\omega(t-s)\eta_\omega(s)\, ds. \tag{49}$$

Differentiating Eq. (49) twice and using Eq. (48) shows that

$$c(b-a)\eta_\omega(t) = \frac{d^2Q_\omega}{dt^2}(t) + (a+b)\frac{dQ_\omega}{dt}(t) + abQ_\omega(t) \tag{50}$$

with

$$Q_\omega(0) = \frac{dQ_\omega}{dt}(0) = 0. \tag{51}$$

In order to simplify the following calculations the assumption will be made that $Q_\omega(\cdot)$ has a simple form, which is defined for $t \geq 0$ and satisfies Eq. (51):

$$Q_\omega(t) = \beta t^2 e^{-\alpha t} \qquad \alpha > 0, \quad \beta > 0. \tag{52}$$

This relationship can be considered as a fitted curve to measured discharge. As before, the parameters α and β are random variables which depend upon the storm ω. Substituting Eq. (52) into Eq. (50) gives

$$c(b-a)\eta_\omega(t) = \beta e^{-\alpha t}[(\alpha-a)(\alpha-b)t^2 + 2(a+b) - 2\alpha)t + 2]. \tag{53}$$

It is noted that if $\alpha \leq a$, then $\eta_\omega(t) > 0$, (for $t \geq 0$); otherwise $\eta_\omega(t)$ is negative for some t. It is also noted that the stochastic process $\eta_\omega(\cdot)$ depends on the random variables a, b, c, α, and β.

Given $\delta > 0$, let $I_\delta = [z, z+\delta]$ denote the δ-time interval of peak area in $e_\omega(\cdot)$. Maximizing the area integral

$$\int_z^{z+\delta} c(e^{-at} - e^{-bt})\, dt, \tag{54}$$

gives z as a function of δ:

$$z = \frac{1}{b-a}\log\left[\frac{1 - e^{-b\delta}}{1 - e^{-a\delta}}\right]. \tag{55}$$

In the last equation, z decreases from $\log(b/a)/(b-a)$ to 0 as δ increases from 0 to ∞. Then $e_\omega^\delta(t) = e_\omega(t)$ for t in I_δ and is zero otherwise, and

$$Q_\omega^\delta(t) = \int_0^t e_\omega^\delta(s)\eta_\omega(t-s)\, ds = \int_z^{z+\delta} c(e^{-as} - e^{-bs})\eta_\omega(t-s)\, ds. \tag{56}$$

This integral can be evaluated exactly, but it suffices to consider that since

$$Q_\omega(t) - Q_\omega^\delta(t) = \int_0^t \{e_\omega(s) - e_\omega^\delta(s)\}\eta_\omega(t-s)\,ds, \tag{57}$$

as δ increases to infinity, the area of $e_\omega(s) - e_\omega^\delta(s)$ goes to zero, and $Q_\omega^\delta(t)$ converges uniformly to $Q_\omega(t)$.

The δ-approximation A_ω^δ to the criterion variable A_ω is given by

$$A_\omega^\delta = \mathrm{A}(Q_\omega^\delta(\cdot)) = \int_0^\infty Q_\omega^\delta(t)\,dt = \int_0^\infty \int_z^{z+\delta} c(e^{-at} - e^{-bt})\eta_\omega(t-s)\,ds dt. \tag{58}$$

This integral can be evaluated, but as δ increases to infinity the inside integral converges uniformly to

$$\int_0^t e_\omega(s)\eta_\omega(t-s)\,ds = Q_\omega(t).$$

The average intensity during peak effective rainfall duration of length δ,

$$\bar{I}(e_\omega^\delta(\cdot)) = \frac{1}{\delta}\int_z^{z+\delta} c(e^{-at} - e^{-bt})\,dt,$$

can be evaluated and simplified using Eq. (55) to obtain

$$\bar{I}(e_\omega^\delta(\cdot)) = \frac{c(b-a)}{\delta ab}\,\frac{[1 - e^{-a\delta}][a/(b-a)]}{[-e^{-b\delta}][b/(b-a)]}. \tag{59}$$

The T-year return frequency value $\bar{e}_T^\delta$ is the T-year return frequency value of the random variable given on the right hand side of Eq. (59). The difference $\Delta e_\omega^\delta(\cdot)$ is given by Eq. (33).

Two storms ω_o and ω, with parameters a_o, b_o, c_o and a, b, c respectively, have the same average synthetic effective rainfall intensity for a given δ if the right hand side of Eq. (59) has the same value for a, b, and c as it does for a_o, b_o, and c_o; and so the set $\Omega_{\omega_o}^\delta$ is the set of all storms ω for which this is true. The structure of this set, and therefore the structure of the conditional expectation

$$E_{\omega_o}^\delta(t) = E[\Delta e_w^\delta(t) \mid \bar{I}(e_{\omega_o}^\delta(\cdot))], \tag{60}$$

depends on the distribution of a, b, and c. Therefore the exact computation of the design storm S_T^δ as given following Eq. (41), also depends on a, b, c, α, and β.

At this point the description of this model depends on the joint distribution of five random variables. In order to reduce the model complexity so that the behavior of the model can be analyzed it is necessary to make further assumptions concerning the relations between these random variables.

The first simplifying assumption is that

$$a = \rho b, \quad 0 < \rho < 1. \tag{61}$$

This further restricts the shape of the effective rainfalls, but in exchange for this restriction Eq. (59) is considerably simplified to:

$$\bar{I}(e_\omega^\delta(\cdot)) = \frac{c(1-\rho)}{\delta ab}\,\frac{[1 - e^{-a\delta}][1/(1-\rho)]}{[1 - e^{-b\delta}][\rho/(1-\rho)]}. \tag{62}$$

The time to peak is $(1/b)\log(1/\rho)/(1-\rho)$, the volume is $c/(b(1-\rho))$, and the peak value of $e_\omega(\cdot)$ is a constant times c. If a later peak implies, on average, a larger volume and a bigger peak, this gives a clue about the possible general relation between b and c. With these facts in mind, one suitable choice, which will also simplify some other equations, is

Table 1. Regression equation coefficients for Orange County, California, rainfall depth duration relationships

return frequency in years	$\log AT$	normal	BT
2	-.610	-.610	.426
5	-.239	-.230	.438
10	-.024	-.043	.427
25	.167	.163	.434
50	.285	.298	.434
100	.397	.418	.427

$$b = 1/kc, \qquad k > 0, \tag{63}$$

with the constant k to be determined later. The effective rainfall pattern chosen this way, with these restrictions put upon it, can be regarded as a possible average among many storms or as a model which only broadly resembles any real effective rainfall curve.

For small δ, the log of the maximum depth of effective rainfall over a time interval of length δ is approximately

$$\log(\delta \bar{I}(e_\omega(\cdot))) \approx \log(1-\rho) + (\rho/(1-\rho))\log\rho + \log\delta. \tag{64}$$

Hydrological data shows that log depth versus log duration curves, for a given return frequency, agree in a general way with this equation; the agreement is that they do plot as straight lines, the disagreement is that the slope of these lines is not necessarily one, as it is in Eq. (64). For example, consider the regression equations for depth $D(\delta)$ versus $\log\delta$,

$$\log D(\delta) = \log AT + BT\log\delta,$$

AT and BT regression coefficients for a given return period T, as determined in Orange County, California (adjusted for time in hours rather than minutes) are shown in Table 1. The values of $\log AT$ are compared with that of a normal distribution with mean -.610 and standard deviation .442.

This table shows that it is plausible to assume that $\log AT$ has a normal $N(\mu,\sigma^2)$ distribution, and in the numerical calculations we will make this assumption and take $\mu = -.610$ and $\sigma = .442$. Equation (64) indicates that we may take

$$\log(c) + \log(1-\rho) + (\rho/(1-\rho))\log\rho = \log AT. \tag{65}$$

Here we are supposing that the result that BT is approximately constant at .43 corresponds to our model result where the slope has the constant value of 1.0, and that the spacing between the log depth curves with return frequency has the same distribution.

Writing a and b in terms of c shows that for a fixed value of t, $e_\omega(t)$ is an increasing function of c. Thus for a fixed value of t, $e_\omega(t)$ is a random variable whose T-year value is obtained by substituting the T-year value for c in the formula for $e_\omega(t)$.

Also it can be shown that $\bar{I}(e_\omega(\cdot))$ is an increasing function of c and so the T-year value of this average intensity is obtained by substituting the T-year value of c into the formula for $\bar{I}(e_\omega(\cdot))$. Thus in this example problem, there is only one storm, the one corresponding to this value of c, with a given intensity and so the T-year design storm consists, in this case, of exactly one storm. And, as noted above, this storm has the property that for each fixed value of t, its value is the T-year value of the random variable $e_\omega(t)$.

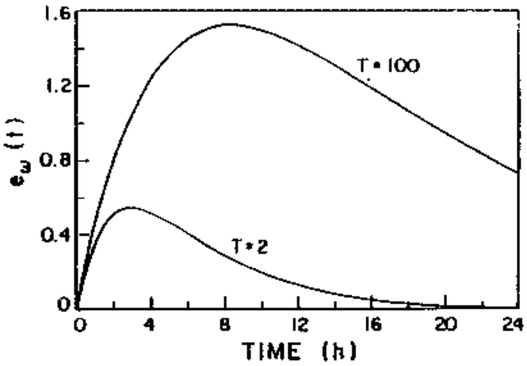


Figure 11

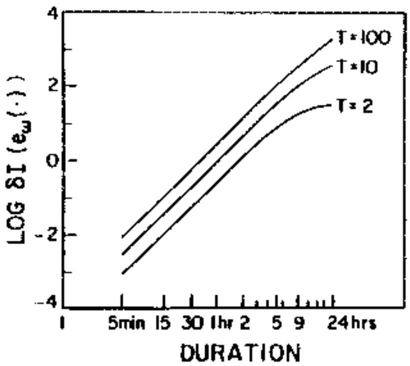


Figure 12

Figure 11. Example problem design storms (effective rainfall) for $T = 2{,}100$ years

Figure 12. Example problem precipitation depth versus duration

Using the above distributions for c, and the relations given between c, a, and b, the T-year design storm can be computed as soon as the values for ρ and k are chosen. The parameter ρ of Eq. (61) is chosen to be $1/2$. The constant k is chosen to be 1 so as to have plausible times to peak effective precipitation (2 year, 3.0 hours; 100 year, 8.3 hours) (see Fig. 11).

With the same parameter choices, $\log \delta \bar{I}(e_\omega(\cdot))$ versus $\log\delta$ curves are given in Fig. 12.

Consider for a given δ the volume of discharge over an interval $[z_o, z_o+\delta]$

$$V_\delta = \int_{z_o}^{z_o+\delta} Q(t)\, dt,$$

which is maximized by the choice

$$z_o = \delta/(e^{\alpha\delta/2} - 1). \tag{66}$$

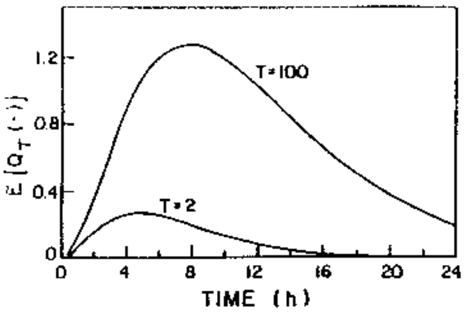
This maximum volume can be written

$$\frac{V_\delta}{\delta} = \frac{2\beta e^{-\alpha z_o}}{\alpha^3} \frac{[\delta(1+\delta z_o) + (2z_o+\alpha z_o^2)]}{(\delta + z_o)^2}. \tag{67}$$

Write $\beta = \gamma/\alpha$, where α and γ are assumed to be independent random variables with $E(\gamma) = \gamma_o$. In order to relate α to c we suppose that if we average over those discharges which come from the same storm then the discharge volume is a fixed multiple ρ_o of the effective precipitation volume

$$(2\gamma_o)/\alpha^4 = \rho_o k c^2. \tag{68}$$

In connection with this averaging, recall that the discharge Q has random variation which is not entirely specified by describing the effective precipitation, so that even exact knowledge of a, b, and c, does not specify α and β and therefore $Q_\omega(\cdot)$ exactly. Consider, for example, the criterion variable which is the total volume of discharge



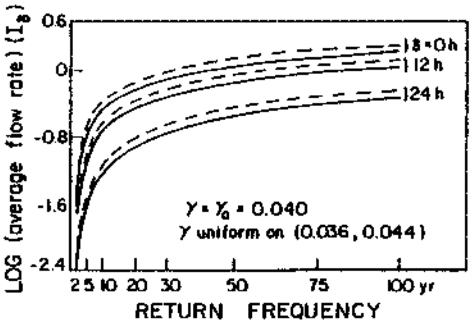


Figure 13

Figure 14

Figure 13. Expected value of $Q_T(\cdot)$ for $T = 2{,}100$ years

Figure 14. Average flow rate during duration I_δ, for both the true distribution (solid) and the design storm approach (dashed)

$$A(Q_\omega(\cdot)) = \int_0^\infty Q_\omega(s)\, ds = 2\beta/\alpha^3. \tag{69}$$

As this simple equation points out, the criterion variable is itself a random variable, whose distribution depends on the joint distribution of α and β. Consequently, for design purposes it is generally not adequate to merely specify one value of the criterion variable but instead a confidence interval for the design value is more appropriate.

We have from Eq. (68)

$$\log\alpha = .25\log(2\gamma_o/\rho_o k) - .5\log c, \tag{70}$$

and so $\log\alpha$ is itself normal since $\log c$ is normal.

The peak value of Q occurs at $2/\alpha$ and is

$$Q_{peak} = 4\beta/e^2\alpha^2 = 4\gamma/e^2\alpha^3.$$

If these peak values are averaged over all storms with the same effective precipitation the average Q_{peak} is $\overline{Q}_{peak} = 4\gamma_o/e^2\alpha^3$. This implies that $\overline{Q}_{peak}$ has a log normal distribution; this property of the model is in good agreement with the usual assumption as to the underlying stream gauge data distribution function.

We choose $\rho_o = 1/2$. The choice $\gamma_o = 0.04$ gives reasonable values for time to peak Q (2 year 4.7 hours, 100 year 7.8 hours) for the distribution of alpha determined by Eq. (70). The resulting discharge curves are shown in Fig. 13.

Figure 14 displays the log of the maximum average discharge over an interval of length δ, as given by Eq. (67), versus return period T, under two different parameter choices. The first choice is in taking β to have the value γ_o/α obtained by averaging over all storms with a given return period T. This can be calculated directly from Eq. (67), which is a decreasing function of α, and the distribution of α. The second choice is in taking β to have the value γ/α, where γ is uniformly distributed on the interval $[.9\gamma_o, 1.1\gamma_o]$; this variation in γ, independent of the effective precipitation, raises the T-year values in the way shown below.

Once the distribution of the parameters a, b, c, α, and β have been described, the realization $\eta_T(\cdot)$ associated with the T-year storm is also completely specified

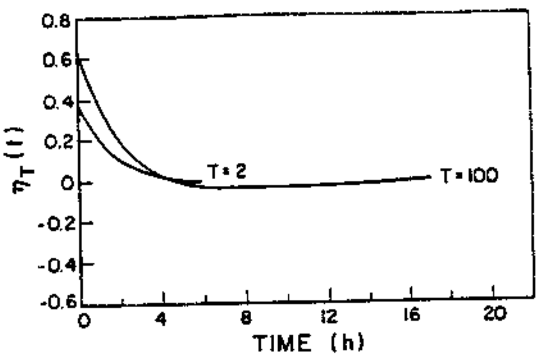


Figure 15. Expected value of the example unit hydrograph for T = 2,100 years

by Eq. (53). The plot in Fig. 15 is of $\eta_T(\cdot)$ for α from the T-year storm and for $\gamma = \gamma_o$. This is the realization which would be obtained by averaging over all such realizations for a given return period T (i.e., using the expected value). The convolution of this average realization with the T-year effective rainfall, is the design storm approach. As Eqs. (49) and (52) show, the true value of Q_ω is $\gamma(\omega)/\gamma_o$ times this estimate, and the error made in this estimation is small only if $\gamma(\omega)$ does not differ much from its expected value γ_o.

9 Conclusions

The well-known design storm approach is developed from a rigorous mathematical analysis of rainfall-runoff data. The mathematical underpinnings of the concept are shown to be well-based in standard probabilistic theory, as applied to the correlation of rainfall-runoff data. For a single rain gage and stream gage data pair, where the peak demand of a criterion variable is of interest at the stream gage site, the standard single area unit hydrograph method with a design storm pattern input of identical return frequencies for all durations is shown to provide the probabilistic distribution of peak demand.

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Technical notes

Evaluation of flow probabilities in run-off detention ponds

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1 Introduction

Stormwater from urban areas may flood into sewers and storm drains at greater volumes than treatment facilities can process, threatening to overwhelm them and cause the release of untreated, contaminated water into local streams. To avoid this, detention basins are built to hold the water until it can be treated. The problem is to build the basins large enough to hold all of the excess water. Only excessively large basins in conjunction with very high treatment capacities would insure that overflow never occurs. However, the cost could be prohibitive. What is needed is a method of determining the probability of overflow for a given basin size and treatment rate. Such information would be useful in evaluating the trade-off of overflow risk and treatment-storage cost.

Design of detention basins and treatment facilities for urban stormwater runoff requires an estimation of the volume, duration and interarrival time of runoff events, as well as the frequency of their occurrence within a given time period. Each of these characteristics may be modeled as a random process. Loganathan et al. (1985) developed an analytical model for computing the probability of overflow of an urban runoff detention basin by considering the randomness of volume, duration, and interarrival time of runoff events. Their model enables the estimation of overflow probability on an event by event basis.