Back to the Unit Hydrograph Method

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ABSTRACT

In this paper, the well-known unit hydrograph method is used as a basis for developing a probabilistic tool for estimating model uncertainty distributions. It is shown that the basic unit hydrograph method includes the uncertainties in the effective rainfall distribution over the catchment, and the uncertainties in interior hydrologic processes (e.g., channel routing, subarea runoff hydrographs, hydrograph summation), that a discretized model (i.e., subdivision of R into subareas or overland flowplanes, linked by channel routing: a link-node model) misrepresents due to the incorrect definition of the effective rainfall distribution over the subareas.

INTRODUCTION

In this paper, the well-known unit hydrograph method is used to recast this modeling approach into a tool for use in developing modeling uncertainty distributions. It is shown that the basic unit hydrograph method includes many of the uncertainties in the effective rainfall distribution over R (that is, the coupled errors due to uncertainties in both rainfall and loss rate estimation), and the uncertainties in interior hydrologic processes (e.g., channel routing, subarea runoff hydrographs, hydrograph summation), that a discretized model (i.e., subdivision of R into subareas or overland flowplanes, linked by channel routing: a link-node model) misrepresents due to the incorrect definition of modeling subarea parameters and processes, and the incorrect definition of the effective rainfall distribution over the subareas.

In the analysis, only one rain gauge and one stream gauge are assumed available for data analysis purposes. In the discretized model, it is generally assumed that the measured rainfalls applies to each subarea. Should the variability in effective rainfall over the catchment be significant then the discretization error can be a major component in the resulting modeling uncertainty.

MODEL ASSUMPTIONS

For probabilistic model development purposes, the following assumptions are used:

1. The catchment R freely drains to a downstream stream gauge, which records the runoff hydrograph, $Q_g{}^i(t)$, for storm event i.
2. The catchment can be represented as a m-subarea link-node model, where for storm i the runoff hydrograph is given by $Q_m{}^i(t)$.
3. The channel system uniformly serves R such that all storms of interest are contained within the channels, with negligible storage effects due to flooding.
4. Channel routing along links is well represented by translation where, for each storm i, there is a "best fit" travel time for the link. (In a latter section, a linear routing technique is used to include the effects of channel storage on flow routing).
5. Runoff hydrographs which confluence can be *represented* as a simple sum.
6. There is a single stream gauge and a single rain gauge available for data analysis purposes.
7. The subarea effective *rainfalls are linear combinations* of the effective rainfall, $e_g{}^i(t)$, measured at the rain gauge site.

For storm event i, the *runoff hydrograph* from subarea j is noted as $Q_j{}^i(t)$. The effective rainfall distribution (rainfall less losses) over subarea j, for storm i, is noted by, $e_j{}^i(t)$. Because it is assumed that for storm i there are characteristic travel times *for translation* channel routing, the runoff hydrograph at the catchment downstream stream gauge, $Q_g{}^i(t)$, equates to the subarea contributions by

$$Q_g{}^i(t) = Q_m{}^i(t) = \sum_{j=1}^{m} Q_j{}^i(t - \tau_j{}^i) \tag{1}$$

where $\tau_j{}^i$ *is* the sum of characteristic travel times for all channel links which connect subarea j to the catchment R stream gauge. Also used in Eq. (1) is the notation that

$$Q_j{}^i(t - \tau_j{}^i) = 0\ , \quad \text{for } t \le \tau_j{}^i \tag{2}$$

It is recalled that in Eq. (1) the $\tau_j{}^i$ may differ for each storm event i, and for each subarea j. Hence, a form of *nonlinearity* is being incorporated into the $Q_m{}^i(t)$ model.

The measured $e_g{}^i(t)$ is seen to be a realization of a stochastic process, as is the stream gauge measured runoff, $Q_g{}^i(t)$. The rainfall-runoff model serves as a correlation between the two stochastic processes. Use of a measured $e_g{}^i(t)$ serves to incorporate rainfall data, loss rates, and antecedent moisture effects.

SUBAREA EFFECTIVE RAINFALL, $e_j{}^i(t)$

In each subarea, R_j, the effective rainfall distribution (i.e., the rainfall less losses) is given by, $e_j{}^i(t)$. In our problem, however, $e_j{}^i(t)$ would be unknown because there is no stream gauge nor rain gauge in subarea j.

By assumption 2, the subarea loss rates are uniform on a subarea basis. However, the precipitation distribution over R_j is unknown. Assumption 7 implies that $e_j{}^i(t)$ is a linear combination of the measured $e_g{}^i(t)$ (at the rain gauge site). Thus,

$$e_j{}^i(t) = \sum_{k=1}^{n_j{}^i} \lambda_{jk}{}^i\, e_g{}^i(t - \theta_{jk}{}^i) \tag{3}$$

where

$n_j{}^i$ = number of proportions of $e_g{}^i(t)$ used to model $e_j{}^i(t)$, from storm i.

$\lambda_{jk}{}^i$ = coefficients for subarea j and storm i; $k = 1,2,\ldots,n_j{}^i$.

$\theta_{jk}{}^i$ = constant timing offsets for subarea j, and storm i, where $e_g{}^i(t - \theta_{jk}{}^i) = 0$ for $t < \theta_{jk}{}^i$.

and all of the $n_j{}^i$, $\lambda_{jk}{}^i$, $\theta_{jk}{}^i$, vary on a storm basis, i.

Use of Eq. (3) is analogous to the unit hydrograph (UH) procedure for developing a runoff hydrograph given an UH and an effective rainfall, except now we develop an effective rainfall in subarea j given an effective rainfall at the rain gauge, $e_g{}^i(t)$, and a set of proportion coefficients. It is assumed that $e_j{}^i(t)$ can be approximated by a formulation using Eq. (3); that is, $e_j{}^i(t)$ is assumed to be linear in $e_g{}^i(t)$.

SUBAREA RUNOFF HYDROGRAPH, $Q_j{}^i(t)$

Given the subarea $e_j{}^i t)$, it is assumed that the "true" runoff hydrograph for storm i, $Q_j{}^i(t)$, is given by the convolution

$$Q_j{}^i(t) = \int_{s=0}^{t} e_j{}^i(t - s)\ \phi_j{}^i(s)\ ds \tag{4}$$

where $\phi_j{}^i(s)$ is a unit hydrograph, for subarea j, and for storm i. It is noted that in Eq. (4), the subarea UH, $\phi_j{}^i(s)$, may differ between storms, i.

Combining Eqs. (3) and (4) gives $Q_j{}^i(t)$ with respect to the available data, $e_g{}^i(t)$, by

$$Q_j{}^i(t) = \int_{s=0}^{t} \sum_{k=1}^{n_j{}^i} \lambda_{jk}{}^i\, e_g{}^i(t - \theta_{jk}{}^i - s)\ \phi_j{}^i(s)\ ds \tag{5}$$

By a change of variables, Eq. (5) can be written as

$$Q_j^i(t) = \int_{s=0}^{t} e_g^i(t - s) \sum_{k=1}^{n_j^i} \lambda_{jk}^i \, \phi_j^i(s - \theta_{jk}^i) \, ds \qquad (6)$$

In Eq. (6), the same UH, $\phi_j^i(s)$, used in Eq. (5) is being used in Eq. (6). Hence, the storm i UH for subarea j is being convoluted with each proportion of $e_g^i(t)$ used to formulate the subarea effective rainfall distribution, $e_g^i(t)$.

REDUCTION OF LINK-NODE MODEL, $Q_m^i(t)$, TO THE SINGLE AREA MODEL, $Q_1^i(t)$

Combining Eqs. (1) and (6) gives the $Q_m^i(t)$ model for the runoff hydrograph at the stream gauge,

$$Q_m^i(t) = \sum_{j=1}^{m} Q_j^i(t - \tau_j^i)$$

$$= \sum_{j=1}^{m} \int_{s=0}^{t} e_g^i(t - s) \sum_{k=1}^{n_j^i} \lambda_{jk}^i \, \phi_j^i(s - \theta_{jk}^i - \tau_j^i) \, ds \qquad (7)$$

For the given assumptions, the measured runoff hydrograph, $Q_g^i(t)$, is correlated to the measured effective rainfall, $e_g^i(t)$, by the model, $Q_m^i(t)$.

The $Q_m^i(t)$ model of Eq. (7) can now be simplified to

$$Q_m^i(t) = \int_{s=0}^{t} e_g^i(t - s) \sum_{j=1}^{m} \sum_{k=1}^{n_j^i} \lambda_{jk}^i \, \phi_j^i(s - \theta_{jk}^i - \tau_j^i) \, ds \qquad (8)$$

or, in a final form

$$Q_m^i(t) = \int_{s=0}^{t} W^i \, e_g^i(t - s) \, \psi^i(s) \, ds \qquad (9)$$

where

$$W^i = \frac{\int_{U=0}^{\infty} \sum_{j=1}^{m} \sum_{k=1}^{n_j^i} \lambda_{jk}^i \, \phi_j^i(U - \theta_{jk}^i - \tau_j^i) \, dU}{\int_{U=0}^{\infty} \sum_{j=1}^{m} \phi_j^i(U - \theta_{jk}^i - \tau_j^i) \, dU} \qquad (10)$$

and

$$\psi^i(s) = \sum_{j=1}^{m} \sum_{k=1}^{n_j^i} \lambda_{jk}^i \, \phi_j^i(s - \theta_{jk}^i - \tau_j^i)/W^i \qquad (11)$$

But the W^i of Eq. (10) can be directly computed by simply equating

$$W^i = \frac{\text{(total runoff volume measured at the stream gauge)}}{\text{(total effective rainfall measured at the rain gauge site)}} \qquad (12)$$

and $\psi^i(s)$ is a standard unit hydrograph. That is, W^i is a constant and $\psi^i(s)$ is the UH, respectively, for storm event, i, given $Q_g^i(t)$ and $e_g^i(t)$.

Consequently the $Q_m^i(t)$ model, for the given assumptions, reduces to simply a single area UH model (without subdivision), noted by $Q_1^i(t)$,

$$Q_m^i(t) = Q_1^i(t) = \int_{s=0}^{t} W^i \, e_g^i(t - s) \, \psi^i(s) \, ds \qquad (13)$$

What is crucial in the formulation of Eq.(13) is that in practice, the "true" m-subarea link-node model, $Q_m^i(t)$, is approximated by the estimator, $\hat{Q}_m^i(t)$, where $\hat{Q}_m^i(t) \neq Q_m^i(t)$ due to following misrepresentations (reference Eq. (7)):

1. Should the $\hat{Q}_m^i(t)$ model be used given only one stream gauge and rain gauge, then $\phi_j^i(s)$ and τ_j^i are estimated by $\hat{\phi}_j^i(s)$ and $\hat{\tau}_j^i$). Whether a subarea UH is used or another runoff approximation such as the kinematic wave overland flow plane, all parameters are estimates only, and are incorrect for storm event i. Only by use of subarea stream gauge and rain gauge data can these parameters be evaluated (i.e., correlated). Additionally, only by use of stream gauge data along the channel links can the τ_j^i be evaluated for storm i. Because subarea and channel link rainfall-runoff data is unavailable, these parameters are all in error.
2. In $\hat{Q}_m^i(t)$, the parameters which reflect the variation in the subarea effective rainfall magnitude (λ_{jk}^i), timing (θ_{jk}^i), and pattern shape $\{n_j^i, \lambda_{jk}^i, \theta_{jk}^i\}$, with respect to the available data, $e_g^i(t)$, are all unknown. Only by use of subarea rainfall-runoff data can these parameters be evaluated for each subarea, and for each storm.

The $Q_1^i(t)$ model UH, $\psi^i(s)$, integrates the following information for storm event i (reference Eq. (11)):

1. The m subarea unknown unit hydrographs for storm i, $\phi_j^i(s)$.
2. The unknown channel link characteristic travel times, summed in the τ_j^i. (Peak attenuation effects are included in a later section.)
3. The summation of runoff hydrographs at confluence points.
4. The unknown distribution of effective rainfall over the entire catchment, R. This includes the unknown effective rainfall distribution magnitude, pattern shape, and timing, with respect to the data, $e_g^i(t)$, measured at the rain gauge site.

THE UNIT HYDROGRAPH, $\psi^i(s)$

For each storm event i, the available data is the pair $\{e_g^i(t), Q_g^i(t)\}$ which are correlated through the $Q_1^i(t)$ model by the $\{W^i, \psi^i(s)\}$. The UH model of Eq. (9) can be rewritten as

$$Q_1^i(t) = \int_{s=0}^{t} e_g^i(t-s)\, W^i\psi^i(s)\, ds$$

$$= \int_{s=0}^{t} e_g^i(t-s)\, \eta^i(s)\, ds \qquad (14)$$

where $\eta^i(s) = W^i\psi^i(s)$ is developed by the correlation of $\{e_g^i(t), Q_g^i(t)\}$ by the standard Voltera integral representation. Consequently, given a set of storms, $\{e_g^i(t)\}$, there must be an associated set of curves, $\{\eta^i(s)\}$, which not only represent the several unknown variations in hydraulic response in R (represented in $Q_m^i(t)$ by the parameters $\phi_j^i(s)$ and τ_j^i), but also the several variations in the effective rainfall distribution over R (represented in $Q_m^i(t)$ by the parameters λ_{jk}^i, θ_{jk}^i, n_j^i, W^i). Because all of these uncertainties and variations cannot be removed without a supply of rainfall-runoff data for each subarea and link used in $Q_m^i(t)$, the modeling output of $Q_1^i(t)$ must be, in a predictive mode, a random variable. Given a design effective rainfall distribution at the rain gauge site of $e_g^D(t)$, then the model output, $Q_1^D(t)$, must be the random variable

$$[Q_1^D(t)] = \int_{s=0}^{t} e_g^D(t)\, [\eta(s)]\, ds \qquad (15)$$

where $[\eta(s)]$ is the distribution of correlation distributions from Eq. (14). In Eq. (15), the brackets are notation for a random variable.

CORRELATING THE RUNOFF HYDROGRAPH TO TWO RAIN GAUGE STATIONS

The development leading to Eqs. (7) and (14) can be extended to the case of the single stream gauge data $(Q_g^i(t))$ being correlated to linear weightings of two effective rainfall measurement sites, noted as $e_{g1}^i(t)$ and $e_{g2}^i(t)$, for storm event i.

In practice, a weighting is usually developed such that a representative effective rainfall is defined by $e_R^i(t)$ where

$$e_R^i(t) = \alpha_1 e_{g1}^i(t) + \alpha_2 e_{g2}^i(t) \qquad (16)$$

such that (usually) $\alpha_1 + \alpha_2 = 1$.

Substituting Eq. (16) into Eq. (7) gives that

$$Q_m^i(t) = \sum_{j=1}^{m} \int_{s=0}^{t} e_R^i(t-s) \sum_{k=1}^{n_j^i} \lambda_{jk}^i\, \phi_j^i(s - \theta_{jk}^i - \tau_j^i)\, ds \qquad (17)$$

where now all parameters $\{\lambda_{jk}^i, \phi_j^i(s), \theta_{jk}^i, \tau_j^i\}$ are determined using the selected $e_R^i(t-s)$ distribution.

From Eq. (15), the single area model $Q_1^i(t)$ still applies, where

$$Q_1^i(t) = \int_{s=0}^{t} e_R^i(t-s)\, \eta^i(s)\, ds \qquad (18)$$

where now the $\eta^i(s)$ correlates the data pair, $\{Q_g^i(t), e_R^i(t)\}$. And in a predictive mode, such as when applying a design storm or hypothetical storm, $e_R^D(t)$, over R, the predictive output, $Q_1^D(t)$, is the random variable

$$[Q_1^D(t)] = \int_{s=0}^{t} e_R^D(t-s)\, [\eta(s)]\, ds \qquad (19)$$

The above discussion applies immediately to three or more effective rainfall measurement sites.

LINEAR ROUTING

In this section, the model development leading to the "true" model $Q_m^i(t)$, is extended to include the effects of unsteady flow routing due to channel storage effects. Let $I_1(t)$ be the inflow hydrograph to a channel flow routing link (number 1), and $O_1(t)$ the outflow hydrograph. A linear routing model of the unsteady flow process is given by

$$O_1(t) = \sum_{k_1=1}^{n_1} a_{k_1}\, I_1(t - \alpha_{k_1}) \qquad (20)$$

where the a_{k_1} are coefficients which sum to unity; and the α_{k_1} are timing offsets. Again, $I_1(t - \alpha_{k_1}) = 0$ for $t < \alpha_{k_1}$. Given stream gauge data for $I_1(t)$ and $O_1(t)$, the best fit values for the a_{k_1} and α_{k_1} can be determined.

Should the above outflow hydrograph, $O_1(t)$, now be routing though another link (number 2), then $I_2(t) = O_1(t)$ and from the above

$$O_2(t) = \sum_{k_2=1}^{n_1} a_{k_2} I_2(t - \alpha_{k_2}) \tag{21}$$

$$= \sum_{k_2=1}^{n_2} a_{k_2} \sum_{k_1=1}^{n_1} a_{k_1} I_1(t - \alpha_{k_1} - \alpha_{k_2})$$

For L links, each with their own respective routing data, the above linear routing technique results in the outflow hydrograph for link number L, $O_L(t)$, being given by

$$O_L(t) = \sum_{k_L=1}^{n_L} a_{k_L} \sum_{k_{L-1}=1}^{n_{L-1}} a_{k_{L-1}} \cdots \sum_{k_2=1}^{n_2} a_{k_2} \sum_{k_1=1}^{n_1} a_{k_1} I_1(t-\alpha_{k_1}-\alpha_{k_2}-\cdots -\alpha_{k_{L-1}}-\alpha_{k_L}) \tag{22}$$

Using vector notation, $O_L(t)$ is written as

$$O_L(t) = \sum_{<k>} a_{<k>} I_1(t - \alpha_{<k>}) \tag{23}$$

For subarea R_j, the runoff hydrograph for storm i, $Q_j{}^i(t)$, flows through L_j links before arriving at the stream gauge and contributing to the total measured runoff hydrograph, $Q_g{}^i(t)$. All of the constants $a_{<k>}$ and $\alpha_{<k>}$ are variable on a storm by storm basis. Consequently from the linearity of the routing technique, the m-subarea link node model is given by the sum of the m contributions, $Q_j{}^i(t)$, where

$$Q_m{}^i(t) = \sum_{j=1}^{m} \sum_{<k>} a^i{}_{<k>_j} Q_j{}^i(t - \alpha^i{}_{<k>_j}) \tag{24}$$

where $<k>_j$ are associated to R_j, and all data is defined for storm i. It is noted that in all cases,

$$\sum_{<k>_j} a^i{}_{<k>_j} = 1 \tag{25}$$

in order to preserve continuity of mass.

LINK-NODE MODEL, $Q_m{}^i(t)$

For the above linear approximations for storm i, Eqs. (3), (6), and (24) can be combined to give the final form for $Q_m{}^i(t)$ (see Hromadka and Whitley,[1]),

$$Q_m{}^i(t) = \sum_{j=1}^{m} \sum_{<k>_j} a^i{}_{<k>_j} \int_{s=0}^{t} e_g{}^i(t - s) \Sigma \lambda_{jk}{}^i \phi_j{}^i(s - \theta_{jk}{}^i - \alpha^i{}_{<k>_j}) ds \tag{26}$$

Because the measured effective rainfall distribution, $e_g{}^i(t)$, is independent of the model, Eq. (26) is rewritten in the form

$$Q_m{}^i(t) = \int_{s=0}^{t} e_g{}^i(t - s) \sum_{j=1}^{m} \sum_{<k>_j} a^i{}_{<k>_j} \Sigma \lambda_{jk}{}^i \phi_j{}^i(s - \theta_{jk}{}^i - \alpha^i{}_{<k>_j}) ds \tag{27}$$

where all parameters are evaluated on a storm by storm basis.

MODEL REDUCTION

The m-subarea model of Eq. (27) can again be directly reduced to the simple single area UH model of Eq. (14) given by $Q_1{}^i(t)$ where

$$Q_1{}^i(t) = \int_{s=0}^{t} e_g{}^i(t - s) \eta^i(s) ds \tag{28}$$

where $\eta^i(s)$ is the correlation distribution between the data pair $\{Q_g{}^i(t), e_g{}^i(t)\}$, given by

$$\eta^i(s) = \sum_{j=1}^{m} \sum_{<k>} a^i{}_{<k>_j} \Sigma \lambda_{jk}{}^i \phi_j{}^i(s - \theta_{jk}{}^i - \alpha^i{}_{<k>_j}) \tag{29}$$

From Eq. (28) it is seen that the classic single area UH model represents a highly complex, rational, link node modeling structure. From Eq. (29), the effects of the uncertainty in the effective rainfall over R, and the uncertainty in the routing parameters, are all properly reflected in the $\eta^i(s)$ which provides the combined distribution of all the considered hydrologic and hydraulic effects. Given a single rain gauge and stream gauge for data correlation purposes, the derived

$\eta^i(s)$ represents the several effects used in the development leading to Eq. (28), integrated according to the several parameters' respective probability distributions. Because the simple $Q_1{}^i(t)$ model structure actually includes most of the effects which are important in flood flow response, it can be used to develop useful probabilistic distributions of modeling output.

CONCLUSIONS

Given a single rain gauge site providing also measured effective rainfall (i.e., rainfall less losses) at the gauge site, and a stream gauge, it is shown that the classic unit hydrograph (UH) method (single area model) represents several hydraulic and hydrologic effects which highly discretized models misrepresent due to the imprecise knowledge of modeling parameters and the unknown effective rainfall distribution over the catchment. The variation in the single area model unit hydrograph represents not only the variations in the catchment hydraulics, but also the important variations in the effective rainfall distribution over the catchment.

The basic single area unit hydrograph model is also shown to represent a highly complex, rational, link-node model which includes (i) variable effective rainfall distributions over each subarea, including variations in distribution magnitude, timing, and storm pattern shape; (ii) variations in the subarea runoff response on a storm by storm basis; (iii) variations in channel flow routing response on a storm by storm basis, including peak flow attenuation and translation timing effects.

Because the basic single area UH model represents a good modeling structure, it can be used to develop hydrologic modeling uncertainty distributions.

REFERENCES:

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