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TECHNICAL NOTE

A TEST FOR ACCURACY OF NUMERICAL SOLUTIONS OF STEADY-STATE HEAT TRANSFER PROBLEMS

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While the number of techniques for numerically approximating steady-state heat transfer problems continues to grow (e.g., domain methods and variants such as finite differences and finite elements, boundary integral methods, and collocation methods), there still is a need for a procedure that develops exact solutions to simplified domain configurations to provide a family of test problems for use in evaluating the merits of a particular technique. In this paper an analytic solution to a family of mixed boundary value problems of the two-dimensional Laplace equation over the unit square is developed. Associated with this family of solutions is an error bound based on a lemma by Hopf. The mathematical development enables the numerical analyst to test the accuracy of other numerical techniques and evaluate the merits of a particular new development or variant of the base method.

INTRODUCTION

Mixed boundary value problems for the Laplacian often arise in engineering problems such as steady-state heat transfer, particularly in the form where the solution u to be found must be harmonic in the interior of a domain, must be a given function on one part of the boundary, and must have normal derivative zero on the remaining part of the boundary. See, for example, [1] for its ability to easily handle such mixed problems. While the number of techniques for numerically approximating steady-state heat transfer problems continues to grow (e.g., domain methods and variants such as finite differences and finite elements, boundary integral methods, and collocation methods), there still is a need for a procedure that develops exact solutions to simplified domain configurations to provide a family of test problems for use in evaluating the merits of a particular technique. That is, it is useful to have sample problems, with computable solutions with precise error bounds, that can be used to test and compare different numerical methods. Such a problem is discussed in the following text. It can easily be solved by the classical method of separation of variables and the explicitly given solution can be computed to a given degree of accuracy. By rotating the domain or interchanging the variables x and y, solutions can be computed for a set of related test problems.

MATHEMATICAL DISCUSSION

The problem is to find $u(x,y)$ so that

$$\nabla^2 u = 0 \qquad 0 < x < 1, 0 < y < 1 \tag{1}$$

NOMENCLATURE

a_n, b_n	series constants	u	potential
du/dn	normal derivative of u	$\nabla^2 u$	Laplacian of u
$g(y)$	specified boundary condition of u on unit square at $x = 1$		

$$u(0, y) = 0 \qquad 0 < y < 1 \tag{2}$$

$$u(x, 1) = 0 \qquad 0 < x < 1 \tag{3}$$

$$u(1, y) = g(y) \qquad 0 < y < 1 \tag{4}$$

$$\frac{du}{dn} = 0 \qquad y = 0, 0 < x < 1 \tag{5}$$

where du/dn denotes the normal derivative of u.

By the method of separation of variables the functions

$$\sinh(\lambda_n x)\cos(\lambda_n y) \qquad \lambda_n = (2n + 1)\frac{\pi}{2}, \; n = 0, 1, \ldots$$

are found that satisfy conditions (1), (2), (3), and (5). These four conditions will also be satisfied by the series

$$\sum b_n \sinh(\lambda_n x)\cos(\lambda_n y) \tag{6}$$

under certain mild hypotheses, and condition (4) will also be satisfied if the Fourier cosine series

$$\sum b_n \sinh(\lambda_n)\cos(\lambda_n y) \tag{7}$$

converges to $g(y)$. In this case

$$a_n = b_n \sinh(\lambda_n) = 2\int_0^1 g(y)\cos(\lambda_n y)\, dy \tag{8}$$

If we suppose that $g(y)$ is continuous with $g(0) = g(1) = 0$ and that g has a derivative g', except possibly at a finite number of points, with $\int_0^1 [g'(x)]^2\, dx$ finite, then the series (6) with coefficients given by (8) converges to the true solution uniformly on the unit square and its boundary. This is established by an argument similar to that given in [2, pp. 81–98], but instead of the maximum principle [2, p. 97] a lemma due to Hopf [3], which applies to the mixed boundary value problem, is needed. The conditions $g(0) = g(1) = 0$ are involved in the proof of the uniform convergence of the series to u on all of the square, and this uniform convergence makes it possible to give uniform error bounds for the approximation of the solution by a partial sum of the series.

One theoretical bound on the difference between the true solution u and the solution u_N obtained by summing up the first N terms of the series above is [2, p. 97]

$$|u(x, y) - u_N(x, y)| \leq AB$$

where

$$A^2 = 2 \int_0^1 [g'(y)]^2 \, dy - \sum_0^N \lambda_n^2 a_n^2$$

and

$$B^2 = \left(\frac{4}{\pi}\right)^2 \left[\frac{\pi^2}{6} - \sum_0^N \frac{1}{(2n + 1)^2}\right]$$

Both A and B go to zero as N tends to infinity. This bound establishes the uniform convergence of u_N to u. A more practical bound, which follows from Hopf's lemma, is

$$|u - u_N| \leq \max\{|u_N(1, y) - g(y)| : 0 < y < 1\} \tag{9}$$

One advantage of the explicit representation of u given by the series (6) with coefficients given by (8) is that various error computations are simplified. For example, in computing $u_N(x, y)$ for $x < 1$, the bound given in Eq. (9) is too large. For a function g bounded in absolute value by M, the nth term in the series (6) is bounded by $2M \sinh(\lambda_n s)/\sinh(\lambda_n)$ and this is approximated closely by

$$2M \exp[-\lambda_n(1 - x)] \tag{10}$$

for x bounded away from 0 and n not too small. Seven of the eight functions given in the program below are bounded by $M = 1$ and the other is bounded by $M = e - 1$; a comparison with the geometric series with terms in Eq. (10) shows that on the grid $x, y = 0.1(0.1)0.9$ thirty terms will give three decimal accuracy. The Fourier series convergence of $u_N(1, y)$ to $g(y)$ on the boundary $x = 1, 0 < y < 1$, is generally much slower than this geometric convergence in the interior.

PROGRAMMING DISCUSSION

A computer program (FORTRAN) was prepared to implement the above development on a personal computer. A copy of the code listing can be obtained from the first author.

The program begins by calling the subroutine "chooseG," which displays eight different choices for the function $g(y)$ to be prescribed on $x = 1$, $0 < y < 1$. The number of the equation chosen is passed to common as the variable NoG. These functions can easily be changed by changing the character variable ge(i), which displays the ith equation, and changing the actual equation for the ith g given in the function "g."

Note that $g(y) = \cos(\lambda_n y)$, which is included only once in the program list of eight functions, for $n = 0$, gives a series (6) with only one term; for large n this provides an interesting test for any numerical method.

In the subroutine "coefficients" 10 terms in the series (6) are obtained by calculating the Fourier cosine coefficients of g, given by Eq. (8), by means of a Simpson's rule integration, which is done in the functions "simpson" and "sumf." In order to use Simpson's rule in the form given, which is for a function of one variable, for the integrand $g(y)\cos(\lambda_n y)$, which is a function of both y and λ_n, the function "gcos" uses the function "g" and the value of λ_n that is passed through common.

The error bound of Eq. (9) is estimated by computing, in the function "errorbound," the maximum of the values $|u_N(1, y) - g(y)|$ for $y = 0.01(0.01)0.99$. With this information, as well as the size of the last Fourier cosine coefficient, the user can decide whether or not to add 10 more terms to the sum of u_N. As the comments in the function "u" indicate, in computing u_N the factor $\sinh(\lambda_n x)/\sinh(\lambda_n)$ will cause overflow unless some care is taken.

The subroutine "output" prints out a comparison of $u_N(1, y)$ with $g(y)$, $y = 0.1(0.1)0.9$, as well as a matrix of the values of $u_N(x, y)$ for $x, y = 0.1(0.1)0.9$. It is also possible to compute the value of $u_N(x, y)$ for input values of x and y.

This program was compiled on an IBM personal computer using the Lahey Computer System's F77L compiler, which, as an extension to FORTRAN 77, allows 31 character variable names.

APPLICATION

Mixed Boundary Value Problem on the Unit Square

The function u is harmonic in the interior; is zero on the sides $x = 0$, $0 < y < 1$ and $y = 1$, $0 < x < 1$; and has zero normal derivative on the side $y = 0$, $0 < x < 1$. On the side $x = 1$, $0 < y < 1$, u is the given function $g(y) = \min(\exp(2y) - 1, \exp(2(1 - y)) - 1)$.

Comparison of g with u on $x = 1$, $0 < y < 1$:

$g(y)$ =	0.221	0.492	0.822	1.226	1.718	1.226	0.822	0.492	0.221
$u(1,y)$ =	0.221	0.492	0.822	1.226	1.682	1.226	0.822	0.492	0.221
y =	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9

The error in approximating the true solution of u is less than 0.367E-01. In the series 30 terms were used.

The values of u are given below on the grid $x, y = 0.1(0.1)0.9$:

$y = 0.1$	0.060	0.121	0.182	0.243	0.302	0.354	0.392	0.399	0.352
$y = 0.2$	0.060	0.120	0.182	0.246	0.311	0.376	0.437	0.487	0.511
$y = 0.3$	0.059	0.118	0.181	0.248	0.321	0.402	0.493	0.598	0.716
$y = 0.4$	0.056	0.114	0.175	0.244	0.322	0.416	0.535	0.695	0.922
$y = 0.5$	0.051	0.105	0.163	0.230	0.309	0.408	0.539	0.725	1.023
$y = 0.6$	0.045	0.092	0.144	0.204	0.276	0.368	0.490	0.659	0.902

$y = 0.7$	0.036	0.074	0.116	0.165	0.225	0.300	0.397	0.521	0.672
$y = 0.8$	0.025	0.052	0.081	0.116	0.158	0.210	0.275	0.351	0.431
$y = 0.9$	0.013	0.027	0.042	0.060	0.082	0.108	0.140	0.175	0.208
	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$	$x = 0.5$	$x = 0.6$	$x = 0.7$	$x = 0.8$	$x = 0.9$

CONCLUSIONS

An analytic solution to a family of mixed boundary value problems for the Laplace equation in two dimensions for a unit square domain is developed. The boundary conditions include zero flux along one side, zero potential (temperature) along two sides, and an arbitrarily specified function for potential along the last side. Because the last side's boundary condition is arbitrary, an infinity of test problems can be developed to test other numerical solutions for the accuracy of the method. Also included is an error bound calculation procedure based on a lemma by Hopf. A computer code listing and program documentation can be obtained from the first author.

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