COMPUTING THE STEADY STATE FREEZING FRONT LOCATION IN TWO-DIMENSIONAL ALGID SOILS

T. V. Hromadka II
Director of Water Resources Engineering
Williamson and Schmid
Irvine, California

C. C. Yen
Hydrologist
Williamson and Schmid
Irvine, California

ABSTRACT

A computer program is developed which calculates the location of the freezing front in a two-dimensional soil system where steady-state temperature boundary conditions are assumed. The mathematical model is based on a new approach to modeling problems which is based upon a generalized Fourier Series for linear operator problems. By using basis functions which satisfy the governing equations for steady-state heat flow (the Poisson or the Laplace equation) with sources or sinks, the solution of the operator relationship over the problem domain is exact; however, the boundary conditions are approximated in the least-squares sense using a newly defined norm which is subsequently minimized to obtian the best approximation. The resulting numerical method is a boundary integral formulation with the added advantage that the error in approximating the boundary conditions is easily seen as a displacement of the true problem boundary. By adding additional basis functions to the approximator or weighting the inner-product (norm) at locations where the problem boundary is highly displaced from the specified position, the boundary condition error of approximation is reduced. Bessel's inequality is used to quantify the error of approximation.

NOMENCLATURE

Ω	Domain, region
$Cl(\Omega)$	Closed domain or region
Γ	Boundary
L	Linear operator
T	Operator
R^m	m-dimensional real space
(u,v)	Inner product of u and v
$\hat{\omega}(z)$	Complex function, i.e., $\hat{\omega}(z) = \hat{\phi}(z) + i\hat{\psi}(z)$
$\hat{\phi}(z)$	Real part of complex function $\hat{\omega}(z)$
$\hat{\psi}(z)$	Imaginary part of complex function $\hat{\omega}(z)$
z	Complex variable, i.e., $z = x+iy$
$\|\|\ \|\|$	Norm
$\cap$	Intersection; $A \cap B$, intersection of A and B
$\cup$	Union; $A \cup B$, union of A and B

INTRODUCTION

In this paper, the mathematical development of the approximation procedure and the application of the technique to a heat transfer problem are presented.

A detailed derivation of the technique and the application to several simple problems are given in Hromadka et al.(6,7). Because this technique shows considerable promise in many engineering applications, the greater computational effort involved over that needed with a finite element or finite difference method solution may be offset by the mathematical attractiveness of a convergence of (in L^2 sense) which can be linearly programmed.

INNER PRODUCTS FOR THE SOLUTION OF LINEAR OPERATOR EQUATIONS

The general setting for solving a linear operator equation with boundary values by means of an inner product is as follows: Let Ω be a region in R^m with boundary Γ and denote the closure of Ω by $cl(\Omega)$. Consider the Hilbert space $L^2(cl(\Omega), d\mu)$, which has inner product $(f,g) = \int fg d\mu$. (This is a real Hilbert space. For the complex version, use the complex conjugate of the function g in the integral.) The way to construct the necessary inner product for the development of a generalized Fourier Series is to choose the measure μ correctly, that is, let μ be one measure μ_1 on Ω and another measure μ_2 on Γ. One natural choice for a plane region would be for μ_1 to be the usual two dimensional Lebesgue measure dV on Ω and for μ_2 to be the usual arc length measure ds on Γ. Then an inner product is given by (Birkhoff and Lynch (9))

$$(f,g) = \int_{\Omega} fg\, dV + \int_{\Gamma} fg\, ds. \qquad (1)$$

Consider a boundary value problem consisting of an operator L defined on domain D(L) contained in $L^2(\Omega)$ and mapping into $L^2(\Omega)$, and a boundary condition operator B defined on a domain D(B) in $L^2(\Gamma)$ and mapping it into $L^2(\Gamma)$. The domains of L and B have to be choosen so at least for f in D(L), Lf is in $L^2(\Omega)$, and for f in D(B),

Bf is in $L^2(\Gamma)$. For example we could have $Lf = \nabla^2 f$, and Bf(s) equal the almost everywhere (a.e.) radial limit of f at the point s on Γ, with appropriate domains.

The next step is to construct an operator T mapping its domain $D(T) = D(L) \cap D(B)$ into $L^2(cl(\Omega))$ by (for example, Davis and Rabinowitz(8))

$$Tf(x) = Lf(x) \text{ for } x \text{ in } \Omega$$
$$Tf(s) = Bf(s) \text{ for } s \text{ on } \Gamma. \tag{2}$$

From (2), there exists a single operator T on the Hilbert space $L^2(cl(\Omega))$ which incorporates both the operator L and the boundary conditions B, and which is linear if both L and B are linear. An application of this procedure using the Complex Variable Boundary Element Method (CVBEM) is given in Hromadka, et al(5). In that study, $Lf = \nabla^2 f$ and Bf is the radial limit of f on Γ. Other applications are contained in Hromadka, et al.(6,7).

Consider the inhomogeneous equation $Lf = g_1$ with the inhomogeneous boundary conditions $Bf = g_2$. Then define a function g on $cl(\Omega)$ by

$$g = g_1 \text{ on } \Omega$$
$$g = g_2 \text{ on } \Gamma$$

Then if the solution exists for the operator equation

$$Tf = g$$

the solution f satisfies $\nabla^2 f = g_1$ on Ω, and $f = g_2$ on Γ in the usual sense of meaning that the radial limit of f is g_2 on Γ. One way to attempt to solve the equation $Tf = g$ is to look at a subspace D_n of dimension n, which is contained in D(T), and to try to minimize $||Th - g||$ over all the h in D_n such as developed in Hromadka, et al.(6,7).

Definition of Inner-Product and Norm

Given a linear operator relationship

$$L\phi = f \text{ on } \Omega, \quad \phi = \phi_b \text{ on } \Gamma \tag{3}$$

defined on the problem domain Ω with auxilliary conditions of $\phi = \phi_b$ on the boundary Γ (see Fig. 1). Here Ω may represent both time and space, and ϕ_b may be both initial and boundary conditions. It is assumed that the working space is sufficiently restricted (see following) such that φ is a unique almost everywhere (ae) solution to (3).

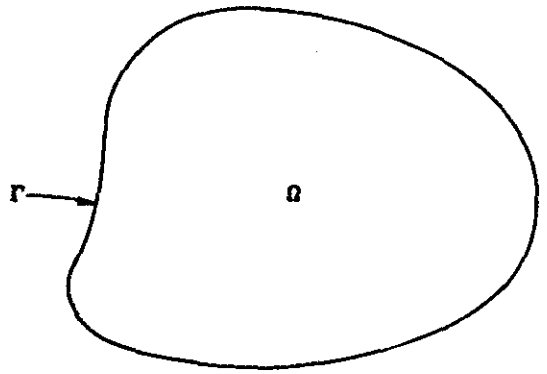

Fig. 1. Definition of Problem Domain, Ω, and Boundary, Γ.

Choose a set of m linearly independent functions $\langle f_j \rangle^m$ (e.g., $\langle 1, x, x^2, \cdots, x^{m-1} \rangle$), and let S^m be the m-dimensional space spanned by the elements of $\langle f_j \rangle^m$. Here, the elements of $\langle f_j \rangle^m$ will be assumed to be functions of the independent variables appearing in (3).

An inner-product is defined for elements of S^m by (u,v) where for $u,v \in S^m$

$$(u,v) = \int_\Gamma uv d\Gamma + \int_\Omega LuLv d\Omega \tag{4}$$

It is seen that (u,v) is indeed an inner-product, because for elements u,v,w in S^m

(i) $(u,v) = (v,u)$
(ii) $(ku,v) = k(u,v)$, for L a linear operator
(iii) $(u+v,w) = (u,w) + (v,w)$ for L a linear operator
(iv) $(u,u) = \int_\Gamma (u)^2 d\Gamma + \int_\Omega (Lu)^2 d\Omega \geq 0$
(v) $(u,u) = 0 \Rightarrow u = 0$ ae on Γ, and $Lu = 0$ ae over Ω

The above restrictions on the operator L imply that L is linear (see (ii) and (iii) in the above definition); if $Lu = 0$ ae over Ω and $u = 0$ ae on Γ, this must imply that the solution $u = [0]$, where [0] is the zero element over $\Omega \cup \Gamma$; and for the inner-product to exist, the integrals must exist. For the inner-product of (4) to exist, the integrands must be finite. Additionally, each element $u \in S^m$ must satisfy $\int_\Gamma u^2 d\Gamma < \infty$.

For the above restrictions of L and the space S^m, the inner-product is defined and a norm "|| ||" immediately follows,

$$||u|| \equiv (u,v)^{\frac{1}{2}} \tag{5}$$

The generalized Fourier series approach can now be used to obtain the "best" approximation $\phi_m \in S^m$ of the function φ using the newly defined inner-product and corresponding norm presented in (4) and (5).

The next step in developing a generalized Fourier series is to construct a new set of functions $\langle g_j \rangle^m$ which are the orthonormal representation of the $\langle f_j \rangle^m$.

Orthonormalization Process

The functions $\langle g_j \rangle^m$ can be obtained by the well-known Gramm-Schmidt procedure (Kantorovich and Krylov(10)) using the newly defined norm of (5). That is,

$$g_1 = f_1/||f_1||$$
$$\vdots$$
$$g_m = [f_m - (f_m,g_1)g_1 - \cdots - (f_m,g_{m-1})g_{m-1}]/$$
$$||f_m - (f_m,g_1)g_1 - \cdots - (f_m,g_{m-1})g_{m-1}|| \tag{6}$$

Hence, the elements of $\langle g_j \rangle^m$ satisfy the convenient properties that

$$(g_j,g_k) = \begin{cases} 0, & \text{if } j \neq k \\ 1, & \text{if } j = k \end{cases} \tag{7}$$

In a subsequent section, a simple one-dimensional problem illustrates the orthonormalization procedure of (6).

The elements $\langle g_j \rangle^m$ also form a basis for S^m but, because of (7), can be directly used in the development of a generalized Fourier series where the computed coefficients do not change as the dimension m of $\langle g_j \rangle^m$

increases. That is, as the number of orthonormalized elements increases in the approximation effort, the previously computed coefficients do not change. Each element $\phi_m \in S^m$ can now be written as

$$\phi_m = \sum_{j=1}^{m} \gamma_j g_j , \quad \phi_m \in S^m \tag{8}$$

where γ_j are unique real constants.

Generalized Fourier Series

The ultimate objective is to find the element $\phi_m \in S^m$ such that $||\phi_m - \phi||$ is a minimum. That is, we want $||\phi_m - \phi||^2$ to be a minimum, where

$$||\phi_m - \phi||^2 = \int_\Gamma \left(\sum_{j=1}^{m} \gamma_j g_j - \phi_b \right)^2 d\Gamma + \int_\Omega \left(L\left(\sum_{j=1}^{m} \gamma_j g_j \right) - L\phi \right)^2 d\Omega \tag{9}$$

Remembering that L is a linear operator, and $L\phi = f$ by the problem definition, of (3) we have that (9) can be rewritten as

$$||\phi_m - \phi||^2 = \int_\Gamma \left(\sum_{j=1}^{m} \gamma_j g_j - \phi_b \right)^2 d\Gamma + \int_\Omega \left(\sum_{j=1}^{m} \gamma_j \, Lg_j - f \right)^2 d\Omega \tag{10}$$

Thus, minimizing $||\phi_m - \phi||^2$ is equivalent to minimizing the error or approximating the boundary conditions and the error of approximating the governing operator relationship in a least-square (or L^2) sense. Because the $\langle g_j \rangle^m$ are orthonormalized and the inner-product is well-defined, the coefficients γ_j of (8) are immediately determined by the generalized Fourier constants, γ_j^*, where

$$\gamma_j^* = (g_j, \phi), \; j = 1,2,\cdots,m \tag{11}$$

Thus

$$\phi_m^* = \sum_{j=1}^{m} \gamma_j^* g_j = \sum_{j=1}^{m} (g_j, \phi) g_j \tag{12}$$

is the "best" approximation of ϕ, in the space S^m.

Because the generalized Fourier series approach is used, several advantages over a matrix solution (for the generalized Fourier series coefficients) are obtained:

1. Elimination of the need for solving large, fully populated, matrices such as occurs when solving the normal system equations.
2. Elimination of the instability which typically arises in a matrix solution for Fourier coefficients (i.e., higher powers of the expansion basis functions assumed).
3. The generalized Fourier series coefficients do not change as additional functions are added (i.e., as the dimension m of the space S^m is increased).
4. Generalized Fourier series theory applies; hence, error analysis can be conducted using Bessel's inequality as discussed in the next section.

Approximation Error Evaluation

Due to the generalized Fourier series approach and the definition of the inner-product, Bessel's inequality applies. That is, for any dimension m

$$(\phi,\phi) \geq \sum_{j=1}^{m} (g_j,\phi)^2 = \sum_{j=1}^{m} \gamma_j^{*2} \tag{14}$$

where

$$(\phi,\phi) = \int_\Gamma (\phi)^2 d\Gamma + \int_\Omega (L\phi)^2 d\Omega = \int_\Gamma \phi^2 d\Gamma + \int_\Omega f^2 d\Omega \tag{15}$$

Equation (15) is readily evaluated and forms an upper bound to the sum of $(g_j,\phi)^2$ as the dimension m increases. Consequently, one may interact with the approximation effort by carefully adding functions to the $\langle f_j \rangle^m$ in order to best reduce the difference computed by Bessel's inequality. The technique of reducing Bessel's inequality can be linearly programmed by choosing additional basis functions which provide the greatest reduction in (14) from the set of basis function available.

APPLICATION TO SLOW MOVING INTERFACE PHASE PROBLEMS

Many freezing/thawing phase change situations fall into the category of heat transfer problems where the heat flux along the phase change boundary is adequately estimated by assuming that the Laplace equation applies. For example, in soil-water phase change in freezing soils, Hromadka and Guymon(1) successfully used the Laplace equation to compute heat flux quantities along a freezing front in order to propagate the front due to soil-water phase change freezing soil column. In another application, Hromadka(4) used the Complex Variable Boundary Element Method or CVBEM to extend the soil-water phase change solution to two-dimensions. A distinct advantage afforded by the CVBEM solution is the error analysis by use of the "approximate boundary" technique (see Hromadka(2)).

In this paper, the CVBEM trial functions a used in order to eliminate the second integral in the inner-product of (4). Thus,

$$(u,v,) = \int_\Gamma uv d\Gamma \tag{16}$$

becomes the inner-product for the generalized Fourier series development.

Modeling Approach

The modeling approach (the governing equations and modeling assumptions are given in Hromadka(3)) starts by developing a CVBEM approximator (Hromadka,(3)) $\hat{\omega}_f(z)$ and $\hat{\omega}_t(z)$ for the frozen and thawed doamins, respectively. The numerical technique determines the analytic function $\hat{\omega}(z)$ which satisfies the boundary conditions of either normal flux or temperature specified at nodal points located on the problem boundary, Γ. Because $\hat{\omega}(z)$ is analytic throughout the interior domain Ω which is enclosed by Γ, then the real and imaginary parts of $\hat{\omega}(z) (= \hat{\phi}(z) + i\hat{\psi}(z)$, where $z = x + iy$) both exactly satisfy the Laplace equation over Ω.

For the quasi-steady state approximation, the governing heat flow equations reduce to the Laplace equations. Consequently, $\hat{\omega}(z)$ determined for both the frozen and thawed regions satisfy the Laplace equations exactly, leaving only errors in satisfying the boundary conditions. To develop a CVBEM steady state solution, an $\hat{\omega}(z)$ is developed for each of the separate regions.

Initially, both $\hat{\omega}_f(z)$ and $\hat{\omega}_t(z)$ are defined by

$$\hat{\omega}_f(z) = \hat{\omega}_f^1, \; z \in \Omega$$
$$\hat{\omega}_t(z) = \hat{\omega}_t^1, \; z \in \Omega \qquad (17)$$

where $\Omega = \Omega_t \cup \Omega_f$ is the global domain, and the first order CVBEM approximators are based on the entire domain. This procedure results in simply estimating the 0°C isotherm location for the homogeneous problem of Ω being entirely frozen or thawed. Let C^1 be the contour corresponding to this 0°C isotherm.

The second iteration step begins by defining Ω_f^2 and Ω_t^2 based on the mutual boundary of C^1. CVBEM approximators $\hat{\omega}_f^2$ and $\hat{\omega}_t^2$ are then defined for Ω_f^2 and Ω_t^2, respectively.

Examining the stream functions $\hat{\psi}_f^2$ and $\hat{\psi}_t^2$, estimates of the discrepancy in matching the flux rates along the interface between Ω_t and Ω_f can be evaluated. The $\hat{\omega}_f^2$ function is now used to determine the next location of the 0°C isotherm. This is accomplished by determining a new $\hat{\omega}_f^2$ with the stream function values of $\hat{\omega}_t^2$ (and modified by conductivity) superimposed at the nodal values of C^1. Next, a new 0°C isotherm C^* is located for $\hat{\omega}_f^*$. The next estimated location for the 0°C isotherm, C^2, is located by averaging the y-coordinates of the nodal points between C^1 and C^*. Figure 2 illustrates this procedure.

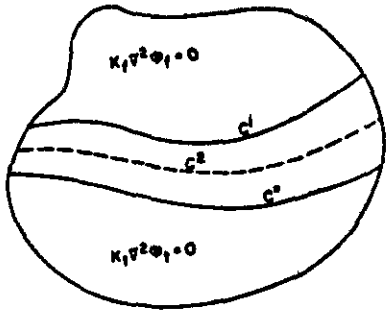

Fig. 2. Iterative Estimation of Freezing Front Location

The third iteration step proceeds by defining Ω_f^3 and Ω_t^3 based on the mutual boundary of C^2 and the above procedure is repeated.

The iteration process continues until the final estimates of Ω_f and Ω_t are determined with corresponding $\hat{\omega}_f$ and $\hat{\omega}_t$ approximators such that

$$|k_f d\hat{\psi}_f/ds - k_t d\hat{\psi}_t/ds| < \delta, \; z \in C \qquad (18)$$

Using the Approximate Boundary

As discussed previously, the subject problem reduces to finding a solution to the Laplace equation in Ω_f and Ω_t coincide along the steady state freezing front location, C. The CVBEM develops approximators $\hat{\omega}_f$ and $\hat{\omega}_t$ which satisfy the the Laplace equation over Ω_f and Ω_t, respectively. Consequently, the only approximation error occurs in matching the boundary conditions continuously on Γ_f, Γ_t, and C. The generalized Fourier series develops the best CVBEM approximation which minimizes the norm in (15), where, due to use of analytic functions as basis functions,

$$(\phi,\phi) = \int_\Gamma \phi^2 d\Gamma \qquad (19)$$

To evaluate the precision in predicting the freezing front location, an approximate boundary is determined for each subproblem domain of Ω_f, Ω_t. The approximate boundary results from plotting the level curves of each CVBEM approximator (i.e. $\hat{\omega}_f$, $\hat{\omega}_t$) which correspond to the boundary conditions of the problem.

Example In Ω_f the thermal boundary conditions for a roadway embankment (Fig. 3) are defined on the problem Γ_f by

ϕ = -10°C, z $\in$ top surface
ϕ = 0°C, z $\in$ freezing front
ψ = 0, z $\in$ left side (symmetry, i.e. zero flux)
ψ = constant, z $\in$ right side (zero flux)

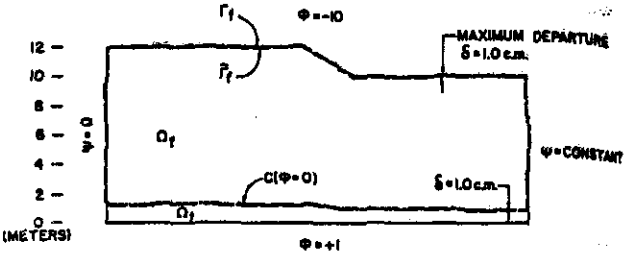

Fig. 3. The Approximate Boundary $\hat{\Gamma}_f$ and the closeness-of-fit to the Problem Boundary, Γ_f

After developing an $\hat{\omega}_f$ and Ω_f from the CVBEM, the approximate boundary $\hat{\Gamma}_f$ is determined by plotting the prescribed level curves. The figure also includes $\hat{\Gamma}_f$ superimposed with Γ_f. Because $\hat{\omega}_f$ is analytic within the are enclosed by the approximate boundary and satisfies the prescribed boundary conditions on the boundary $\hat{\Gamma}_f$, then $\hat{\omega}_f$ is the exact solution of the boundary value problem redefined on $\hat{\Gamma}_f$ and its interior, $\hat{\Omega}_f$. Should $\hat{\Gamma}_f$ completely cover Γ_f, then $\hat{\omega}_f$ is the exact solution to the subject problem.

Thus, the CVBEM modeling error is directly evaluated by the closeness-of-fit between $\hat{\Gamma}_f$ and Γ_f. However in this application, the approximate boundary concept is used not only to examine the closeness-of-fit to the boundary conditions, but possibly more crucial, the closeness-to-fit of matching the estimated freezing front location between Ω_f and Ω_t along the contour, C. Should Ω_f and Ω_t match C continuously, then $\hat{\omega}_f$ and $\hat{\omega}_t$ equate thermal flux continuously along C.

Applications

Figure 3 depicts an application of the geothermal model for a roadway embankment problem and the use of the approximate boundary. Figure 4 illustrates the two-dimensional steady state freezing front location on a geothermal problem involving a buried subfreezing 3-meter diameter pipeline. An examination of the approximate boundaries indicate that a good CVBEM approximator was determined by use of a 26-node CVBEM model. The maximum departure δ between the approximate boundaries and the problem boundary Γ occurred along the top of the pipeline and had a value of approximately 3.5 cm. The average departure $\bar{\delta}$ is estimated at less than 1 cm. The freezing front maximum departure is approximately 4 cm and occurred at the problem's right-hand side. Average departure on C is less than 2 cm.

The example problems presented illustrate the usefulness of the CVBEM in predicting the quasi-steady state freezing front location for two-dimensional problems. Possibly the most important result is the accurate determination of the approximation error involved in using the CVBEM. The usual procedure in estimating the freezing front is to use a finite element or finite difference numerical analog. A hybrid of these

domain methods is to include a variable mesh in order to better accommodate the interface. However, none of these methods provide the error of approximation. In comparison, the CVBEM model provides the approximation error not only in matching the boundary conditions, but in predicting the interface location between Ω_f and Ω_t. And this error is simple to interpret as an approximate boundary displacement from the true problem boundary, and the displacement between Ω_f and Ω_t along the 0°C isotherm contour, C.

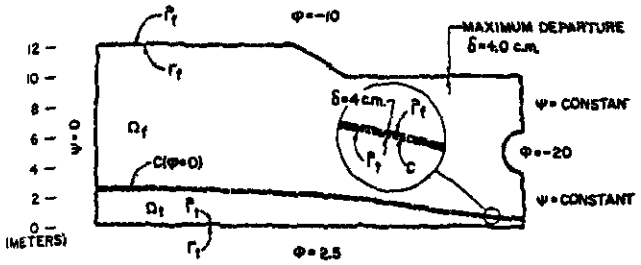

Fig. 4. Application of the CVBEM Geothermal Model to Predict Steady-State Conditions

Time-stepped Approximate Boundary

By plotting the several CVBEM generated approximate boundaries, the time evolution of approximation error is readily seen. From Fig. 5, it is concluded that the computational effort employed by the CVBEM analysis is adequate for this case study. The figure shows a variation in the approximate boundary location as the solution progresses in time; however, the variation is of less than 1.0 cm in magnitude.

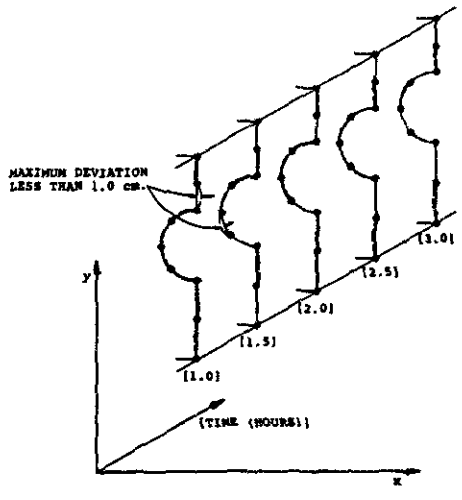

Fig. 5. Approximate Boundary Evolution for Time-stepped Problem Solution (see Fig. 4 for Domain Definition)

Conclusions

In this paper, the CVBEM basis functions are utilized in a generalized Fourier series which uses the inner-product of (15) to determine the basis function coefficients. Because analytic functions are used, the inner-product reduces to a least-square fit of the boundary conditions. The CVBEM is used to approximate a slowly-moving interface between two quasi-potential problem solutions. The approximate boundary technique is used to demonstrate the CVBEM modeling error in achieving the prescribed boundary conditions as the time-stepped advancement in time is approximated.

References

1. Hromadka II, T. V. and Guymon, G. L., "Simple Model of Ice Segregation Using an Analytic Function to Model Heat and Soil-Water Flow," ASME Third Int. Symposium on Offshore Mechanics and Artic Engg., New Orleans, LA (1984).
2. Hromadka II, T. V., "The Complex Variable Boundary Element Method: Development of Approximative Boundaries," Engineering Analysis, Vol. 1, No. 4, (1984), pp. 218-222.
3. Hromadka II, T. V., "The Complex Variable Boundary Element Method," Springer-Verlag (1984).
4. Hromadka II, T. V., "Predicting Two-Dimensional Steady-State Freezing Fronts using the CVBEM and an Approximative Boundary," Journal of Heat Transfer, ASME, in-press (1986).
5. Hromadka II, T. V. and Yen, C. C., "A Complex Boundary Element Model of Flow-Field Problems Without Matrices," Engineering Analysis (1986) in-press.
6. Hromadka II, T. V., Pinder, G. F. and Joos, B., "Approximating a Linear Operator Equation Using a Generalized Fourier Series: Development," Journal of Computational Mechanics, (1986) in-review.
7. Hromadka II, T. V., Yen, C. C. and Pinder, G. F., "Approximating a Linear Operator Equation Using a Generalized Fourier Series: Applications," Journal of Computational Mechanics, (1986) in-review.
8. Davis, P. J. and Radinowitz, P., Advances in Orthonormalizing Computation, Academic Press, 1961.
9. Birkhoff, G., and Lynch, R., "Numerical Solution of Elliptic Problems," SIAM Studies in Applied Math, 1984.
10. Kantorovich, L. V. and Krylov, V. I., "Approximate Methods of Higher Analysis," Interscience Publishers, New York (1964).