WATER RESOURCES RESEARCH, VOL. 17, NO. 5, PAGES 1425–1430, OCTOBER 1981

Nodal Domain Integration Model of Unsaturated Two-Dimensional Soil-Water Flow: Development

T. V. HROMADKA II, G. L. GUYMON, AND G. C. PARDOEN

Department of Civil Engineering, University of California, Irvine, California 92717

The nodal domain integration method is applied to a two-dimensional unsaturated soil water flow problem where the solution domain is discretized into irregular triangular elements and the state variable is approximated by a spatial linear trial function within each triangular element. The resulting element matrices incorporate the well-known Galerkin finite element, subdomain, and integrated finite difference numerical statements as special cases of the nodal domain integration numerical statement.

INTRODUCTION

Numerical solutions of two-dimensional nonlinear partial differential equations such as those that occur in the theory of unsaturated ground water flow are generally limited to solution by the finite difference or finite element methods. Finite difference approximations, such as those described by *Spalding* [1972], can be derived for rectangular and also for irregular two-dimensional domains. Finite element methods [*Pinder and Gray,* 1977] can also be applied to irregular two-dimensional domains. Both methods are often compared to each other for numerical 'efficiency' or other descriptions of superiority [*Hayhoe,* 1978].

Recently, *Hromadka and Guymon* [1980*a, b, c*] have developed a new numerical approach called the nodal domain integration method, which has been applied to one-dimensional linear and nonlinear problems. From this numerical model, the finite difference, subdomain, and Galerkin finite element methods are included in a single numerical statement.

In this paper, the nodal domain integration method is applied to the two-dimensional triangular finite element. As special cases, the Galerkin finite element, subdomain, and finite difference numerical models are determined by the appropriate specification of a single parameter in the resulting nodal domain integration numerical statement. Thus all three numerical approaches are included in one numerical statement similar to the usual Galerkin finite element matrix system.

The purpose of this paper is twofold. The first objective is to present a basic description of the nodal domain integration procedure as applied to the class of partial differential equations generally encountered in the theory of unsaturated groundwater flow. Detailed mathematical derivations and applications of this numerical approach for a one-dimensional problem are contained in other papers [*Hromadka and Guymon,* 1980*b*, *c*]. The theoretical foundations of this numerical method are based on the well-known subdomain technique of the finite element weighted residuals approach. The second objective is to develop a numerical statement which represents the finite element Galerkin statement, subdomain numerical statement, finite difference integrated control volume statement, and the nodal domain integration statement by the specification of a single parameter in the resulting triangle element matrix system.

GOVERNING EQUATIONS

Two-dimensional unsaturated Darcian soil water flow in a nondeformable homogeneous porous media is assumed described by the partial differential equation

$$\frac{\partial}{\partial x}\left(K_h \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_h \frac{\partial \phi}{\partial y}\right) = \frac{\partial \theta}{\partial t} \qquad (x, y) \in \Omega \tag{1}$$

$$K_h = K_h(x, y, \psi, t) \tag{2}$$

where K_h are anisotropic hydraulic conductivity values in the (x, y) directions, respectively, ϕ is the total hydraulic energy head ($\phi = \psi + y$), ψ is the soil water pore pressure head, and θ is the volumetric water content. In (1), water content is assumed to be a single valued function of soil water pore pressure according to the usual soil drying curve with hysterisis effects neglected. Thus

$$\begin{aligned} \theta &= \theta(\psi) \qquad \psi < 0 \\ \theta &= \theta_0 \qquad \psi \geq 0 \end{aligned} \tag{3}$$

where θ_0 is assumed constant. A volumetric water content to pore pressure gradient is defined by

$$\begin{aligned} \theta^* &= \frac{\partial \theta}{\partial \psi} \qquad \psi < 0 \\ \theta^* &= 0 \qquad \psi \geq 0 \end{aligned} \tag{4}$$

For the above assumptions, (1) is rewritten as

$$\frac{\partial}{\partial x}\left[K_h \frac{\partial \phi}{\partial x}\right] + \frac{\partial}{\partial y}\left[K_h \frac{\partial \phi}{\partial y}\right] = \theta^* \frac{\partial \phi}{\partial t} \qquad (x, y) \in \Omega \tag{5}$$

NODAL DOMAIN DISCRETIZATION OF SOLUTION DOMAIN

Consider the partial differential operation

$$A(\phi) = f \qquad (x, y) \in \Omega \qquad \Omega = \Omega \cup \Gamma \tag{6}$$

with boundary condition types of Dirichlet or Neumann specified on boundary Γ. An m-nodal point distribution can be defined in Ω with arbitrary density (Figure 1) such that an approximation $\hat{\phi}$ for ϕ is defined in Ω by

$$\hat{\phi} = \sum_{j=1}^{m} N_j(x, y)\phi_j \qquad (x, y) \in \Omega \tag{7}$$

Paper number 1W0810.
0043-1397/81/001W-0810$01.00

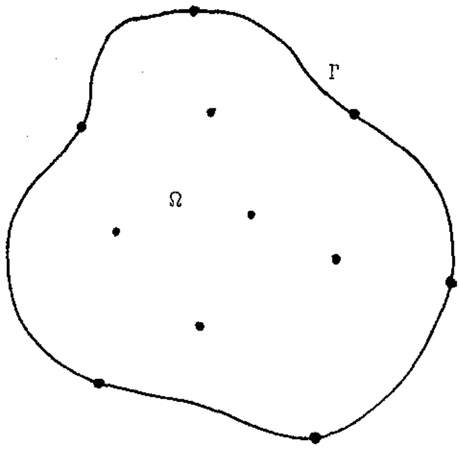

Fig. 1. Distribution of nodal points in two-dimensional domain Ω with boundary Γ.

where $N_j(x, y)$ are linearly independent global shape functions and ϕ_j are assumed values of the state variable ϕ at nodal point j. In (7) it is assumed that

$$\lim_{m\to\infty} \hat{\phi} = \lim_{\max\|(x_j,y_j),(x_k,y_k)\|\to 0} \hat{\phi} = \phi \qquad (x, y) \in \Omega \tag{8}$$

The nodal domain integration approach uses the topology of sets resulting from the discretization procedure associated with the well known finite element and integrated finite difference methods. Generally, the global domain Ω is discretized into finite elements or control volumes and subdomains, depending on whether the finite element or integrated finite difference approach is used. These two discretizations share a common nodal domain discretization of Ω, consequently, the resulting numerical approximations form the various numerical methods which can be defined by a single unifying analog. In the following a subdomain R_j and a finite element $\hat{\Omega}_e$, a discretization of the global domain Ω, is defined. From these two set covers of Ω a unifying nodal domain cover of Ω is defined. A closed connected spatial subset R_j is defined for each nodal point j such that

$$\Omega = \bigcup_{j=1}^{m} R_j \tag{9}$$

with supplementary conditions of

$$(x_j, y_j) \in R_j \qquad (x_j, y_j) \notin R_k \qquad j \neq k \tag{10}$$

$$R_j = R_j \cup B_j \tag{11}$$

where (x_j, y_j) are the spatial coordinates of node j and B_j is the boundary of R_j. It is assumed that every subdomain is disjoint except along shared boundaries, i.e.,

$$R_j \cap R_k = B_j \cap B_k \tag{12}$$

The subdomain method of the finite element weighted residuals approach approximates (6) by solving the m equations

$$\int_{\Omega} (A(\phi) - f) w_j \, dA = 0 \tag{13}$$

where

$$\begin{aligned} w_j &= 1 \qquad (x, y) \in R_j \\ w_j &= 0, \qquad (x, y) \notin R_j \end{aligned} \tag{14}$$

A second cover of Ω is defined by the finite element method with

$$\Omega = \cup \, \hat{\Omega}_e \tag{15}$$

where $\hat{\Omega}_e$ is the closure of finite element domain Ω_e and its boundary $\hat{\Gamma}_e$.

Let S_e be the set of nodal points defined by

$$S_e \equiv \{j | \hat{\Omega}_e \cap R_j \neq \{0\}\} \tag{16}$$

Then a set of nodal domains Ω_j is defined for each finite element domain $\hat{\Omega}_e$ by

$$\Omega_j = \hat{\Omega}_e \cap R_j \qquad j \in S_e \tag{17}$$

The subdomain method of weighted residuals as expressed by (13) can be rewritten in terms of the subdomain cover of Ω by

$$\int_{\Omega} (A(\phi) - f) w_j \, dA = \int_{R_j} (A(\phi) - f) \, dA \tag{18}$$

With respect to the finite element discretization of Ω,

$$\int_{R_j} (A(\phi) - f) \, dA = \int_{R_j \cap \hat{\Omega}_e} (A(\phi) - f) \, dA \tag{19}$$

where for each finite element domain $\hat{\Omega}_e$

$$\int_{\hat{\Omega}_e \cap R_j} (A(\phi) - f) \, dA = \int_{\Omega_j} (A(\phi) - f) \, dA \qquad j \in S_e \tag{20}$$

From the above subset definitions and set covers of Ω, application of the usual subdomain method to the governing partial differential operation of (6) is accomplished by an integration of the governing equations over the nodal domains interior of each finite element, resulting in a finite element matrix system similar to that determined by the Galerkin finite element method. The spatial definition of each nodal domain Ω_j depends on the definition of both the finite element and subdomain covers of Ω and is therefore somewhat arbitrary. A convenient criterion is to define the nodal domains such that the resulting finite element matrix system is symmetric. This symmetry property is used for the definition of finite element nodal domains in the following model development of two-dimensional unsaturated soil water flow.

NODAL DOMAIN INTEGRATION MODEL

The operator relationship for the two-dimensional unsaturated soil water flow model of (5) is

$$A(\phi) - f = \frac{\partial}{\partial x}\left[K_h \frac{\partial \phi}{\partial x}\right] + \frac{\partial}{\partial y}\left[K_h \frac{\partial \phi}{\partial y}\right] - \theta^* \frac{\partial \phi}{\partial t} \tag{21}$$

Substituting (21) into (20) gives the finite element matrix system for $\hat{\Omega}_e$ (Figure 3)

$$\left\{\int_{\Omega_j} \left(\frac{\partial}{\partial x}\left[K_h \frac{\partial \phi}{\partial x}\right] + \frac{\partial}{\partial y}\left[K_h \frac{\partial \phi}{\partial y}\right] - \theta^* \frac{\partial \phi}{\partial t}\right) dA\right\} = \{0\} \qquad j \in S_e \tag{22}$$

Expanding (22) gives

$$\left\{\int_{\Gamma_j \cap \Gamma_e} \left(K_h \frac{\partial \phi}{\partial n}\right)\bigg|_{\hat{\Gamma}_e} ds\right\} + \left\{\int_{\Gamma_j - \Gamma_j \cap \hat{\Gamma}_e} \left(K_h \frac{\partial \phi}{\partial n}\right)\bigg|_{\Gamma_j} ds\right\} = \left\{\int_{\Omega_j} \theta^* \frac{\partial \phi}{\partial t} \, dA\right\} \qquad j \in S_e \tag{23}$$

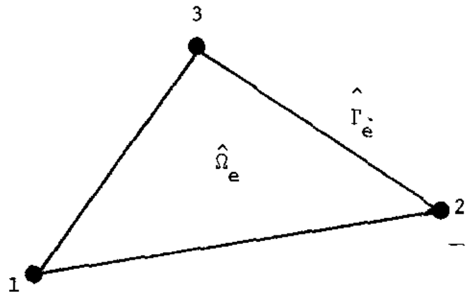

Fig. 2. Subdomain R_j as the union of all nodal domains associated to nodal point j.

where the first term of (23) cancels due to flux contributions from neighboring finite elements (Figure 2) or satisfies zero-flux natural boundary conditions on Γ and where (n, s) are normal and tangential vector components on B_j, Γ_j, and $\hat{\Gamma}_e$. The finite element discretization of Ω is assumed to be composed of triangles with three vertex-located nodal points associated to each finite element domain $\hat{\Omega}_e$ (Figure 3).

Integration of the governing flow equation on each Ω_j involves the definition and integration of nonlinear parameters K_h and θ^*. *Hromadka and Guymon* [1980*c*] expand the nonlinear parameters by Taylor series and integrate the expanded infinite series expression resulting in an equivalent numerical approximation as a function of the assumed trial function nodal point values. Another approach to handling the nonlinearity problem is to approximately linearize the governing flow equation by assuming the nonlinear parameters to be uniform in the finite element [*Myers*, 1971] for small durations of time, Δt. Some methods of determining quasiconstant values for nonlinear parameters are examined for the one-dimensional unsaturated soil water flow problem by *Hromadka and Guymon* [1980*a*]. Using quasiconstant values of $(K_h^{(e)}, \theta^*(e))$ for the nonlinear parameters of the governing flow equation for a small time step Δt simplifies the finite element matrix system of (23) to

$$\left\{\int_{\Gamma_j-\Gamma_j\cap\Gamma_e}\left(K_h^{(e)}\,\frac{\partial\phi}{\partial n}\right)\bigg|_{\Gamma_j} ds\right\} = \left\{\theta^*(e)\,\frac{\partial}{\partial t}\int_{\Omega_j}\phi\,dA\right\} \quad (24)$$
$$0 \le t \le \Delta t \qquad j \in S_e$$

The nodal domain integration method solves (24) for each $\hat{\Omega}_e$ by defining functions for a Δt timestep:

$$\int_{\Omega_j}\phi\,dA = \int_{\Omega_j} f_j(\phi_k, t)\,dA \qquad \{j, k \in S_e\} \quad (25)$$

$$\int_{\Gamma_j-\Gamma_j\cap\hat{\Gamma}_e}\left(\frac{\partial\phi}{\partial n}\right)\bigg|_{\Gamma_j} ds = c_j(t)\int_{\Gamma_j-\Gamma_j\cup\Gamma_e}\left(\frac{\partial\bar{\phi}}{\partial n}\right)\bigg|_{\Gamma_j} ds \qquad j \in S_e \quad (26)$$

where $f_j(\phi_k, t)$ is a function of time and finite element domain $\hat{\Omega}_e$ associated nodal points; correction factor $c_j(t)$ is a function of time; and $\bar{\phi}$ is a linear trial function for ϕ in $\hat{\Omega}_e$. The above function definitions are extensions of a similar set of function definitions determined for a one-dimensional soil water flow problem [*Hromadka and Guymon*, 1981]. In the study of one-dimensional problems it was concluded that the $f_j(\phi_k, t)$ functions had a far greater effect on model accuracy than did the $c_j(t)$ functions and that the simplifying definition

$$c_j(t) = 1 \quad (27)$$

could be made for many problems. This conclusion is valid for both first- and second-order polynomial trial functions where the finite element discretization is composed of nodal domains satisfying the matrix symmetry criterion.

For the assumed triangular finite element discretization of Ω, a definition of nodal domains Ω_j is required in order to evaluate the $f_j(\phi_k, t)$ functions. Using matrix symmetry as a criterion, element nodal domains are defined by the intersection of triangle finite element medians (Figure 4) partitioning the triangle into three equal areas. The definition of $f_j(\phi_k, t)$ used for each nodal domain Ω_j is

$$f_j(\phi_k, t) = \left[\frac{(\eta_j(t)\phi_j + \sum_{k\neq j}\phi_k)}{2 + \eta_j(t)}\right]\frac{A^{(e)}}{3} \qquad (j, k) \in S_e \quad (28)$$

where $A^{(e)}$ is the area of triangle $\hat{\Omega}_e$. In order to provide element matrix symmetry,

$$f_j(\phi_k, t) \equiv f(\bar{\eta}(t), \phi_k) \qquad (j, k) \in S_e \quad (29)$$

where

$$\bar{\eta}(t) = \frac{1}{3}\sum\eta_j(t) \qquad j \in S_e \quad (30)$$

For finite element domain $\hat{\Omega}_e$ the above gives the element capacitance $\mathbf{P}^{(e)}$ matrix approximation

$$\mathbf{P}^{(e)}[\bar{\eta}(t)] = \frac{\theta^*(e)A^{(e)}}{3(\bar{\eta}(t) + 2)}\begin{bmatrix}\bar{\eta}(t) & 1 & 1\\ 1 & \bar{\eta}(t) & 1\\ 1 & 1 & \bar{\eta}(t)\end{bmatrix} \quad (31)$$

For $c(t) = 1$, the element conduction matrix $\mathbf{K}^{(e)}$ for $\hat{\Omega}_e$ is determined from (24) and (26). From (26) the state variable flux term $\partial\phi/\partial n$ is approximated on $(\Gamma_j - \Gamma_j \cap \hat{\Gamma}_e)$ by assuming ϕ to be described in $\hat{\Omega}_e$ by a linear trial function.

In order to evaluate the spatially integrated flux terms of (26) for each nodal domain of a triangular finite element, the triangle geometry is defined by a system of vectors as shown in Figure 5. For the assumed linear trial function variation of the state variable ϕ in the finite element triangle, the spatially integrated flux term contribution to nodal domain Ω_1 is geometrically determined by Figure 6. Flux must contribute to Ω_1 through the boundaries of Ω_1 and can be calculated by the flux vector through state variable ϕ values ϕ_1 (at node 1) and ϕ' as shown in the figure where

$$\phi' = \frac{1}{L}(\phi_2 d_1 + \phi_3 d_2) \quad (32)$$

The integration of the spatial boundary of Ω_1 normal to the considered flux vector is $L/2$ as shown in Figure 6. From

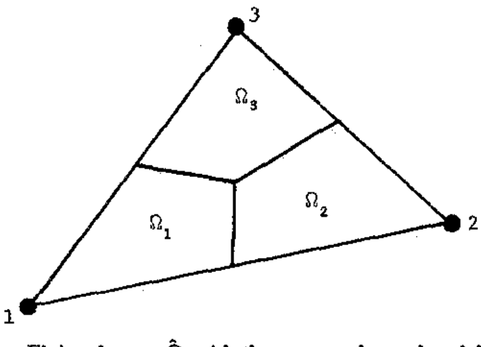

Fig. 3. Finite element $\hat{\Omega}_e$ with three vertex-located nodal points.

Darcy's Law, the efflux for a linear polynomial function approximation is

$$-\frac{K_h^{(e)}[\phi_2(d_1/L)+\phi_3(d_2/L)-\phi_1]}{h} \tag{33}$$

and the integrated efflux (discharge) contribution from Ω_1 is

$$-\frac{K_h^{(e)}}{2hL}[\phi_2 d_1 L+\phi_3 d_2 L-L^2\phi_1] \tag{34}$$

which is obtained by multiplying $L/2$ with (33). From Figure 5 the geometric constants in (34) are

$$L^2=\vec{r}_{23}\vec{r}_{23}=x_{23}{}^2+y_{23}{}^2 \tag{35}$$

$$d_1L=\vec{r}_{13}\vec{r}_{23}=x_{13}x_{23}+y_{13}y_{23} \tag{36}$$

$$d_2L=-\vec{r}_{12}\vec{r}_{23}=-(x_{12}x_{23}+y_{12}y_{23}) \tag{37}$$

where $x_{23}=x_3-x_2$. Using matrix notation, (34) may be written as follows:

$$\frac{K_h^{(e)}}{4A^{(e)}}[x_{23}{}^2+y_{23}{}^2),\ -(x_{13}x_{23}+y_{13}y_{23}),\ (x_{12}x_{23}+y_{12}y_{23})]\begin{Bmatrix}\phi_1\\ \phi_2\\ \phi_3\end{Bmatrix} \tag{38}$$

Combining the finite element nodal domain equations, the element conduction matrix $\mathbf{K}^{(e)}$ for Ω_e is

$$\mathbf{K}^{(e)}=\frac{K_h^{(e)}}{4A^{(e)}}\begin{bmatrix}(x_{23}{}^2+y_{23}{}^2), & -(x_{13}x_{23}+y_{13}y_{23}), & (x_{12}x_{23}+y_{12}y_{23})\\ (\text{symmetric}) & (x_{13}{}^2+y_{13}{}^2), & -(x_{12}x_{13}+y_{12}y_{13})\\ & & (x_{12}{}^2+y_{12}{}^2)\end{bmatrix} \tag{39}$$

The approximation of (24) by the nodal domain integration element matrix system for Ω_e is

$$\mathbf{K}^{(e)}\phi_j+\mathbf{P}^{(e)}[\bar{\eta}(t)]\,\dot{\phi}_j=\{0\} \qquad j\in S_e \tag{40}$$

where ϕ_j and $\dot{\phi}_j$ are the vector of nodal point values and time derivative of nodal point values associated to finite element domain Ω_e.

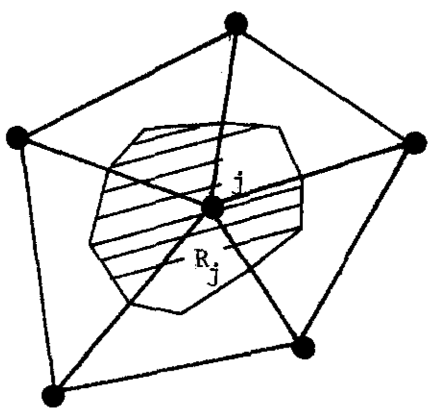

Fig. 4. Finite element partitioned into nodal domains.

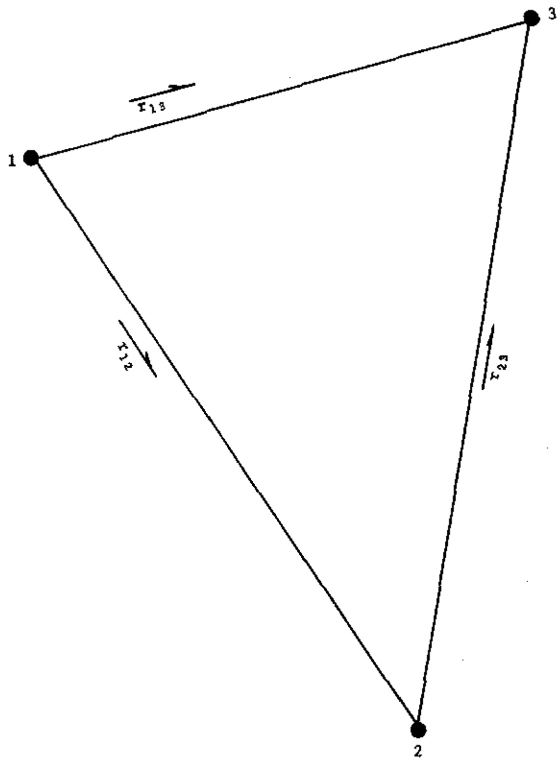

Fig. 5. Vector description of triangle finite element geometry.

Similarity of Nodal Domain Integration Model to Other Numerical Models

In this section the finite element subdomain and Galerkin techniques of the weighted residuals method [*Pinder and Gray*, 1977] and the integrated finite difference method as developed by *Spalding* [1972] will be applied to the assumed linearized soil water flow equation. The models derived from these numerical approaches will be compared to the nodal domain integration model and an appropriate $\bar{\eta}(t)$ determined such that the element matrix system of (40) also represents these other various modeling approaches.

Integrated Finite Difference Method

By using a control volume defined by the union of all nodal domains associated to a particular nodal point (Figure 2), the integrated finite difference approach can be derived. The control volume CV_j is defined by

$$CV_j=\cup\Omega_j \tag{41}$$

The integrated influx to the control volume along the boundary is the sum of influx contributions from each interior nodal domain Ω_j. The nodal domain Ω_j efflux contribution from CV_j (by means of the boundary Γ_j) is determined from (38). The total integrated efflux from CV_j would be row j of the assembled global conduction matrix derived by the usual sum of element conduction matrices of (39).

The integrated finite difference model assumes that ϕ is constant-valued in CV_j. Consequently,

$$\int_{CV_j}\phi\,dA\equiv\phi_j\int_{CV_j}dA \tag{42}$$

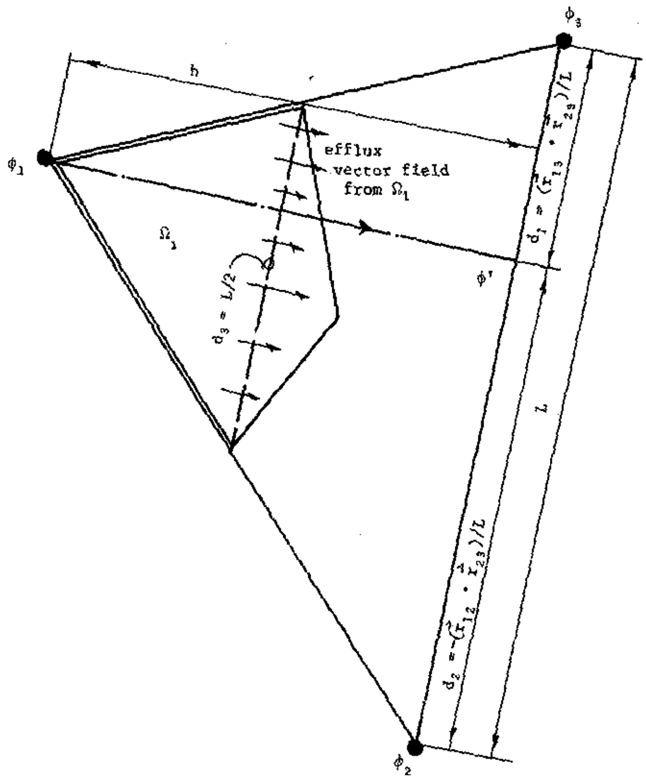

Fig. 6. Geometric solution of flux distribution (for nodal domain Ω_1) for assumed linear trial function distribution of state variable.

Holding ϕ constant in each Ω_j gives

$$\int_{\Omega_j} \phi \, dA = \phi_j \int_{\Omega_j} dA = \phi_j \frac{A^{(e)}}{3} \tag{43}$$

The element capacitance matrix of (31) includes the integrated finite difference statement of (43) by

$$\lim_{\bar{\eta}(t)\to\infty} \mathbf{P}^{(e)}[\bar{\eta}(t)] = \frac{\theta^*(e)A^{(e)}}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{44}$$

Therefore the integrated finite difference model for the linearized soil water flow problem is

$$\lim_{\bar{\eta}(t)\to\infty} (\mathbf{K}^{(e)} \phi_j + \mathbf{P}^{(e)}[\bar{\eta}(t)] \dot{\phi}_j = \{0\}) \qquad j \in S_e \tag{45}$$

Subdomain Method of Weighted Residuals

A subdomain model for the assumed linearized soil water flow problem can be derived from the nodal domain integration model by prescribing the trial function $\hat{\phi}$ to be linear in each finite element domain Ω_e. Using a linear trial function $\hat{\phi}$ in Ω_e allows a direct integration of $\hat{\phi}$ in each Ω_j (Figure 7). Therefore a subdomain approximation in Ω_1 is

$$\int_{\Omega_1} \hat{\phi} \, dA = \frac{A^{(e)}}{108} [22\phi_1 + 7\phi_2 + 7\phi_3] \tag{46}$$

By comparison to the integrated finite difference model,

$$R_j \equiv CV_j \tag{47}$$

Therefore the element capacitance matrix $\mathbf{P}^{(e)}[\bar{\eta}(t)]$ includes a subdomain model

$$\mathbf{P}^{(e)} \left[\frac{22}{7}\right] = \frac{\theta^*(e)A^{(e)}}{108} \begin{bmatrix} 22 & 7 & 7 \\ 7 & 22 & 7 \\ 7 & 7 & 22 \end{bmatrix} \tag{48}$$

The integrated efflux from subdomain R_j through boundary B_j is given by the jth row of the assembled global conduction matrix. Therefore a subdomain model for the linearized soil water flow problem is

$$\left\{\mathbf{K}^{(e)} \phi_j + \mathbf{P}^{(e)} \left[\frac{22}{7}\right] \dot{\phi}_j = \{0\}\right\} \qquad j \in S_e \tag{49}$$

Galerkin Method of Weighted Residuals

The Galerkin finite element approach applied to the linearized soil water flow problem for triangular elements and a linear trial function [*Myers*, 1971] is included in the nodal domain integration model by

$$(\mathbf{K}^{(e)}\phi_j + \mathbf{P}^{(e)}[2] \dot{\phi}_j = \{0\}) \qquad j \in S_e \tag{50}$$

Nodal Domain Integration Method

From the above the nodal domain integration model includes the integrated finite difference, subdomain, and Galerkin finite element models for constant values of $\bar{\eta}(t) = (2, 22/7, \infty)$. Consequently, a computer model based on the nodal domain integration element matrix systems also includes the above numerical models by the specification of a single constant for $\bar{\eta}(t)$. These results can be compared to the one-di-

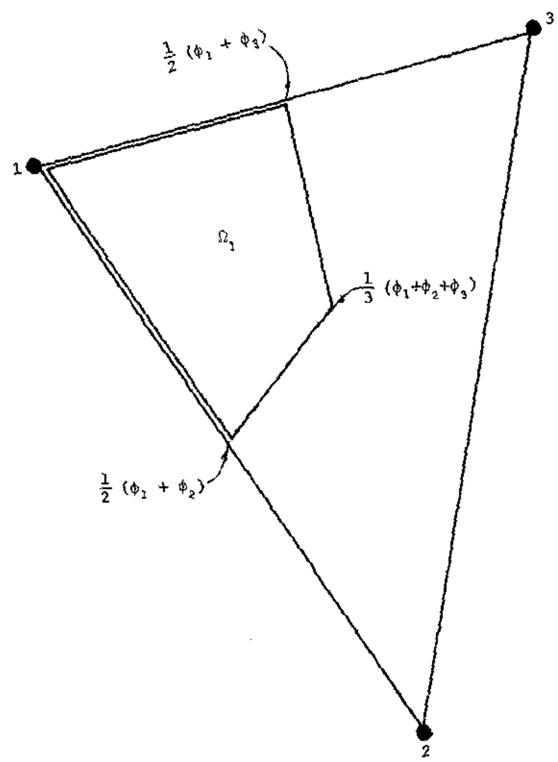

Fig. 7. Linearly distributed state variable values in nodal domain partition Ω_1.

mensional nodel domain integration model [*Hromadka and Guymon,* 1980*a*] which represents the finite element, subdomain, and finite difference models for $\eta = (2, 3, \infty)$, respectively.

CONCLUSIONS

The nodal domain integration numerical approach has been used to determine a numerical analog which incorporates the Galerkin finite element, subdomain, and integrated finite difference methods as special cases. The resulting numerical statement involves the same computational requirements as does the Galerkin finite element procedure. Thus computer programs may be prepared based on the nodal domain integration procedure which inherently contains the Galerkin finite element, subdomain, and finite difference techniques. A powerful method of comparing the accuracy of various numerical techniques is provided which eliminates uncertainty of effects between codes used for comparison.

Theoretically, the so-called 'nodal domain integration' method contains all numerical subsets in addition to those derived; i.e., finite element, finite difference, and subdomain methods. For instance, this method would include linear basis function approximations of higher order basis functions. The method proposed here can be extended to include the case where a single computational problem can be allowed to select a spatially and temporarily varying η function to achieve optimal spatial accuracy.

Acknowledgments. This work was supported by the U.S. Army Research Office (Research Grant DAAG24-79-C-0080). The second author was on sabbatical leave at the U.S. Army Cold Regions Research and Engineering Laboratory, Hanover, N. H. during the preparation of this paper.

REFERENCES

Hayhoe, H. N., Study of relative efficiency of finite difference and Galerkin techniques for modeling soil-water transfer, *Water Resour. Res., 14*(1), 97–102, 1978.

Hromadka, T. V., II, and G. L. Guymon, Some effects of linearizing the unsaturated soil moisture transfer diffusion model, *Water Resour. Res., 16*(4), 643–650 1980*a*.

Hromadka, T. V., II, and G. L. Guymon, Numerical mass balance for soil-moisture transfer problems, *Advan. Water Resour., 3,* 107, 1980*b*.

Hromadka, T. V., II, and G. L. Guymon, Note on time integration of soil moisture transport, *Advan. Water Resour., 3,* 181–186, 1980*c*.

Hromadka, T. V., II, and G. L. Guymon, Improved linear shape function model of soil moisture transport, *Water Resour. Res., 17*(3), 504–512, 1981.

Myers, G. E., *Analytical Methods in Conduction Heat Transfer,* McGraw-Hill, New York, 1971.

Pinder, G. F., and W. G. Gray, *Finite Element Simulation in Surface and Subsurface Hydrology,* Academic, New York, 1977.

Spalding, D. B., A novel finite-difference formulation for differential expressions involving both first and second derivations, *Int. J. Numer. Methods Eng., 4,* 551, 1972.

(Received November 24, 1980;
revised April 27, 1981;
accepted May 5, 1981.)