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Improved Linear Trial Function Finite Element Model of Soil Moisture Transport

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Two methods of modeling a higher-order approximation function of soil moisture transport by an improved linear trial function approximation are presented. The first approach considered is based upon use of the alternation theorem and a finite element capacitance matrix that incorporates the Galerkin finite element, subdomain, finite difference, and proposed nodal domain integration methods. The second approach extends the first approach by developing a temporal relationship for element matrices such that a higher-order approximation function can be modeled by a linear approximation function. Comparison of model results produced from a nodal domain integration model incorporating these improved linear trial function approximations to the finite element, subdomain, and finite difference methods indicates that this approach may lead to a generalized modeling method for soil moisture transport problems.

Introduction

The study of numerical methods for the approximation of linear and nonlinear soil moisture transport in a one-dimensional domain has received some recent attention. *Hayhoe* [1978] compared the numerical effectiveness between the finite element and finite difference numerical methods in modeling a sharp wetting front soil infiltration problem. A special finite difference analog was advanced as the best numerical approach to the problem studied. *Hromadka and Guymon* [1980*a*] further studied the sharp wetting front problem and developed a modification to the finite element method which resulted in an increase in model accuracy for a linear soil water diffusivity problem. For a nonlinear diffusivity problem the traditional finite element formulation gave comparable results to *Hayhoe*'s [1978] finite difference approach when the finite element analog used constant element diffusivity values as determined by a spatial estimation procedure. A procedure to determine which numerical (domain) method to use for simulation of all moisture transport problems, however, is not advanced.

In this paper, two approaches for increasing numerical model accuracy by modeling a higher-order or a more complex family of trial functions by linear trial functions are presented. Such modeling procedures would benefit from the lower computational effort associated with smaller matrix arrays and yet incorporate some of the increase in numerical accuracy usually provided by higher-order trial function approximation sets.

By use of the alternation theorem for determining a 'best' approximation of a lower-order polynomial estimator to a higher-order polynomial or function, an adjustment error distribution is determined which is a function of the discretized domain nodal point set. Use of this error distribution function in a subdomain integration procedure aids in incorporating some benefits of a higher-order approximation function set into a lower-order approximation function set.

Another possibility is to define appropriate correction factors (as a function of time) which equate the various soil moisture transport terms as modeled by a family of higher-order trial functions to the first-order trial function model approximations. Like the alternation theorem approach, the resulting numerical model has the reduced matrix computer memory requirements but incorporates some of the benefits of a higher-order trial function approximation set.

We call this extension of the subdomain version of the finite element weighted residuals method the 'nodal domain integration method.' For the class of problems considered, the resulting element matrix system determined from the nodal domain integration procedure is a function of a single parameter η, which may be variable with respect to both space and time. Thus η may vary between finite elements and also change as the numerical simulation progresses in time. As special cases of the nodal domain integration element matrix system, specified constant values of η correspond to the Galerkin finite element, subdomain, and finite difference approximations. Consequently, a computer algorithm based on the resulting element matrix system derived from the nodal domain integration method will also represent these other specified numerical approaches for certain specified values of the parameter η.

Mathematical Development

The one-dimensional horizontal soil moisture transport model for an unsaturated soil column is

$$\frac{\partial}{\partial x}\left[D\frac{\partial\theta}{\partial x}\right] = \frac{\partial\theta}{\partial t} \qquad x \in \Omega$$
$$\Omega \equiv \{x|0 \le x \le L\} \tag{1}$$

where θ is the volumetric water content (θ less than the soil's porosity), x is the spatial coordinate, t is time, D is the soil water diffusivity and is a function of soil water content, and Ω is the spatial domain of definition.

The domain Ω can be discretized by n nodal points into n disjoint subdomains:

$$\begin{aligned}\Omega_1 &\equiv \{x|0 \le x \le (x_1 + x_2)/2\}\\ \Omega_2 &\equiv \{x|(x_1 + x_2)/2 < x \le (x_2 + x_3)/2\}\\ &\vdots\\ \Omega_n &\equiv \{x|(x_{n-1} + x_n)/2 < x \le x_n = L\}\end{aligned} \tag{2}$$

where x_j is the spatial coordinate associated to nodal point value θ_j and

$$\Omega = \bigcup_{j=1}^{n} \Omega_j \tag{3}$$

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Equation (1) must be satisfied on each Ω_j. Therefore n equations are generated by solving

$$\frac{\partial}{\partial x}\left[D\frac{\partial\theta}{\partial x}\right] = \frac{\partial\theta}{\partial t} \qquad x \in \Omega_j \tag{4}$$

where

$$D = D(\theta) \qquad \theta = \theta(x, t) \tag{5}$$

Integrating (4) with respect to space gives

$$\left\{D\frac{\partial\theta}{\partial x}\right\}\Big|_{\Gamma_j} = \frac{\partial}{\partial t}\int_{\Omega_j}\theta\,dx \qquad x \in \Omega_j \tag{6}$$

where Γ_j is the spatial boundary of region Ω_j. Integrating (6) with respect to time gives

$$\int_{k\Delta t}^{(k+1)\Delta t}\left\{D\frac{\partial\theta}{\partial x}\right\}\Big|_{\Gamma_j} dt = \int_{\Omega_j}\{\theta\}\Big|_{\Gamma_t} dx \tag{7}$$

where Γ_t is the limits of temporal integration between time steps $k\Delta t$ and $(k+1)\Delta t$. Equation (7) can be simplified by using the linear transformation

$$t = k\Delta t + \epsilon \qquad 0 \le \epsilon \le \Delta t \tag{8}$$

Thus

$$\int_0^{\Delta t}\left\{D(k\Delta t + \epsilon)\frac{\partial\theta(k\Delta t + \epsilon)}{\partial x}\right\}\Big|_{\Gamma_j} d\epsilon = \int_{\Omega_j}\{\theta\}\Big|_{\Gamma_t} dx \tag{9}$$

The soil water diffusivity function can be expressed with respect to time by the Taylor series

$$D(x = x_0, k\Delta t + \epsilon) = \sum_{i=0}^{\infty}\frac{D^{(i)}(x = x_0, k\Delta t)\epsilon^i}{i!} \tag{10}$$

where (i) is the ith order temporal partial differential operator, and x_0 is a specified spatial coordinate. Combining (9) and (10) gives

$$\int_0^{\Delta t}\left\{\left(\sum_{i=0}^{\infty}\frac{D^{(i)}(k\Delta t)\epsilon^i}{i!}\right)\frac{\partial\theta(k\Delta t + \epsilon)}{\partial x}\right\}\Big|_{\Gamma_j} d\epsilon = \int_{\Omega_j}\{\theta\}\Big|_{\Gamma_t} dx \tag{11}$$

For a spatial local coordinate system defined by

$$y \equiv \{x | 0 \le y \le l_j\} \qquad x \in \Omega_j$$
$$dy = dx \tag{12}$$
$$l_j = (x_{j+1} - x_{j-1})/2$$

(11) can be expanded as

$$\sum_{i=0}^{\infty}\frac{D^{(i)}(y = l_j, k\Delta t)}{i!}\int_0^{\Delta t}\epsilon^i\left\{\frac{\partial\theta(k\Delta t + \epsilon)}{\partial x}\right\}\Big|_{y=l_j} d\epsilon - \sum_{i=0}^{\infty}\frac{D^{(i)}(y = 0, k\Delta t)}{i!}\int_0^{\Delta t}\epsilon^i\left\{\frac{\partial\theta(k\Delta t + \epsilon)}{\partial x}\right\}\Big|_{y=0} d\epsilon = \int_{\Omega_j}\{\theta\}\Big|_{\Gamma_t} dx \tag{13}$$

The soil water content function is approximated spatially and temporally by

$$\theta(x, t) \simeq \hat\theta(x, t)$$

$$\hat\theta(x, t) = \sum_{r=1}^{n} N_r(x)\left(\sum_{m=0}^{k+1} M_m(t)\theta_r^m\right) \tag{14}$$

where N_r and M_m are the linearly independent spatial and temporal shape functions and

$$\theta_r^m = \hat\theta(x = x_r, t = m\Delta t) \tag{15}$$

where the θ_r^m are known values for time steps $m = \{0, 1, \cdots, k\}$ and x_r is the spatial coordinate of node r. The spatial gradient of the soil water content function is approximated by

$$\frac{\partial\theta}{\partial x} \simeq \frac{\partial\hat\theta}{\partial x} = \sum_{r=1}^{n}\frac{\partial N_r}{\partial x}\left(\sum_{m=0}^{k+1} M_m\theta_r^m\right) \tag{16}$$

Substituting (14) and (16) into (13) gives the numerical approximation of the governing flow equation in Ω_j:

$$\sum_{i=0}^{\infty}\frac{D^{(i)}(y = l_j, k\Delta t)}{i!}\int_0^{\Delta t}\epsilon^i\left\{\sum_{r=1}^{n}\frac{\partial N_r}{\partial x}\left(\sum_{m=0}^{k+1} M_m\theta_r^m\right)\right\}\Big|_{y=l_j} d\epsilon - \sum_{i=0}^{\infty}\frac{D^{(i)}(y = 0, k\Delta t)}{i!}\int_0^{\Delta t}\epsilon^i\left\{\sum_{r=1}^{n}\frac{\partial N_r}{\partial x}\left(\sum_{m=0}^{k+1} M_m\theta_r^m\right)\right\}\Big|_{y=0} d\epsilon = \int_0^{l_j}\left\{\sum_{r=1}^{n} N_r\left(\sum_{m=0}^{k+1} M_m\theta_r^m\right)\right\}\Big|_{\Gamma_t} dy \tag{17}$$

The unknown values of nodal points θ_r^{k+1} can be solved by equating

$$\sum_{i=0}^{\infty}\frac{D^{(i)}(y = l_j, k\Delta t)}{i!}\cdot\int_0^{\Delta t}\epsilon^i\left\{\sum_{r=1}^{n}\frac{\partial N_r}{\partial x}M_{k+1}\theta_r^{k+1}\right\}\Big|_{y=l_j} d\epsilon - \sum_{i=0}^{\infty}\frac{D^{(i)}(y = 0, k\Delta t)}{i!}\cdot\int_0^{\Delta t}\epsilon^i\left\{\sum_{r=1}^{n}\frac{\partial N_r}{\partial x}M_{k+1}\theta_r^{k+1}\right\}\Big|_{y=0} d\epsilon - \int_0^{l_j}\sum_{r=1}^{n}N_r\theta_r^{k+1}\,dx$$
$$= \sum_{i=0}^{\infty}\frac{D^{(i)}(y = 0, k\Delta t)}{i!}\int_0^{\Delta t}\epsilon^i\left\{\sum_{r=1}^{n}\frac{\partial N_r}{\partial x}\cdot\left(\sum_{m=0}^{k} M_m\theta_r^m\right)\right\}\Big|_{y=0} d\epsilon - \sum_{i=0}^{\infty}\frac{D^{(i)}(y = l_j, k\Delta t)}{i!}\int_0^{\Delta t}\epsilon^i\left\{\sum_{r=1}^{n}\frac{\partial N_r}{\partial x}\cdot\left(\sum_{m=0}^{k} M_m\theta_r^m\right)\right\}\Big|_{y=l_j} d\epsilon - \int_0^{l_j}\sum_{r=1}^{n}N_r\theta_r^k\,dx \tag{18}$$

APPROXIMATION IMPROVEMENT BY USE OF ALTERNATION THEOREM (NODAL DOMAIN INTEGRATION METHOD)

The space-time surface approximated by (14) can be simplified by assuming that the functional surface $\theta(x, t)$ can be described by sets of piecewise continuous functions. For a first-order polynomial spatial trial function approximation $\hat\theta$ for θ between nodal point values $(\theta_{j-1}, \theta_j, \theta_{j+1})$,

$$\frac{\partial\hat\theta}{\partial x}\Big|_{y=l} = (\theta_{j+1} - \theta_j)/l$$

$$\frac{\partial\hat\theta}{\partial x}\Big|_{y=0} = (\theta_j - \theta_{j-1})/l$$

$$\int_{y=0}^{l} \theta\, dy = \frac{l}{8}\{\theta_{j-1} + 6\theta_j + \theta_{j+1}\} \tag{19}$$

where for discussion purposes it is assumed that in (19),

$$x_{j+1} - x_j = x_j - x_{j-1} = l$$
$$dx = dy$$

A major problem in the linear spatial approximation of θ between neighboring nodal points is that θ curvature is not modeled. Thus a higher-order polynomial approximation or a more complex family of trial functions may be useful. However, additional computer memory is usually required due to the increase in the resulting matrix bandwidth incorporating additional nodal points (degrees of freedom) in the higher-order approximation. Another possibility is to approximate the more complex or higher-order trial function approximations $\hat{\theta}$ for θ with linear trial functions $\bar{\theta}$. That is, determine the best linear approximation $\bar{\theta}$ to $\hat{\theta}$ between consecutive nodal points. For example, let

$$\hat{\theta} = \theta_j + \frac{(\theta_{j+1} - \theta_j)}{\sin(\pi/2\alpha)} \sin\left(\frac{\pi Z}{2l'\alpha}\right) \qquad Z \in \hat{\Omega} \tag{20}$$

where a local coordinate Z is defined by

$$\Omega = x | x_j \le x \le x_{j+1}$$
$$\hat{\Omega} = Z | 0 \le Z \le l' \tag{21}$$

and

$$l' = x_{j+1} - x_j$$
$$\alpha \ge 1 \tag{22}$$

The best linear approximation $\bar{\theta}$ for $\hat{\theta}$ on $\hat{\Omega}$ is given from the alternation theorem [*Cheney*, 1966, p. 75] by setting

$$\bar{\theta}(0) - \hat{\theta}(0) = +e \tag{23a}$$
$$\bar{\theta}(\mu) - \hat{\theta}(\mu) = -e \tag{23b}$$
$$\bar{\theta}(l') - \hat{\theta}(l') = +e \tag{23c}$$

where e equals a constant error and

$$\bar{\theta}(Z) = \lambda Z + \beta \qquad 0 \le Z \le l' \tag{24}$$
$$0 < \mu < l'$$

Solution of condition (23*a*) gives

$$\bar{\theta}(0) - \hat{\theta}(0) = \beta - \theta_j = +e \tag{25}$$

Solution of condition (23*c*) gives

$$\bar{\theta}(l') - \hat{\theta}(l') = \lambda l' + \beta - \theta_{j+1} = +e \tag{26}$$

From (25) and (26),

$$\bar{\theta}(Z) = \frac{(\theta_{j+1} - \theta_j)}{l'} Z + \theta_j + e \qquad Z \in \hat{\Omega} \tag{27}$$

Solving for e, (23*b*) is differentiated and set to zero:

$$\frac{d}{dZ}(\bar{\theta}(Z) - \hat{\theta}(Z)) = 0 \qquad Z \in \hat{\Omega} \tag{28}$$

giving

$$\mu = \frac{2l'\alpha}{\pi} \cos^{-1}\left[\frac{2\alpha \sin(\pi/2\alpha)}{\pi}\right] \tag{29}$$

Combining (23*b*) and (29) gives

$$e = \frac{(\theta_{j+1} - \theta_j)}{2\pi}\left\{\frac{(\pi^2 - 4\alpha^2 \sin^2(\pi/2\alpha))^{1/2}}{\sin(\pi/2\alpha)} - 2\alpha \cos^{-1}\left[\frac{2\alpha \sin(\pi/2\alpha)}{\pi}\right]\right\} \tag{30}$$

In (30), $\alpha = 1$ corresponds to a quarter cycle of the general sinusoidal curve, whereas $\alpha \to \infty$ corresponds to a zero-curvature approximation (straight line). For a given value of $\alpha = \alpha_0$,

$$e(\theta_j, \theta_{j+1}, \alpha_0) = e(\theta_j, \theta_{j+1}) \tag{31}$$

Therefore for $Z \in \hat{\Omega}$

$$\bar{\theta}(Z) = (\theta_{j+1} - \theta_j)\frac{Z}{l'} + \theta_j + e(\theta_j, \theta_{j+1}) \tag{32}$$

Comparison of (32) to (19) indicates that the spatial gradient terms remain similar, but the integration of $\bar{\theta}$ differs from that of θ in (19) due to the e term. Thus anagolous to (19),

$$\left.\frac{\partial\bar{\theta}}{\partial x}\right|_{y=l} = (\theta_{j+1} - \theta_j)/l$$
$$\left.\frac{\partial\bar{\theta}}{\partial x}\right|_{y=0} = (\theta_j - \theta_{j-1})/l \tag{33}$$
$$\int_{y=0}^{l} \bar{\theta}\, dy = \int_{y=0}^{x_j} \bar{\theta}\, dy + \int_{y=x_j}^{l} \bar{\theta}\, dy$$

The selection of the approximation in (20) is arbitrary. Another possibility is to fit a polynomial to all nodal values in Ω and solve for $e(\theta_j, \theta_{j+1})$ for each Δt time step.

Time Integration Approximation

For Δt time steps a linear polynomial function approximation may be used to model the time variation of $\theta(x, t)$ between time steps $(k, k + 1)$, where $(k + 1)$ is the time step to be evaluated; thus

$$\theta_j(k\Delta t + \epsilon) = (\theta_j^{k\Delta t})\left(\frac{\Delta t - \epsilon}{\Delta t}\right) + (\theta_j^{(k+1)\Delta t})\frac{\epsilon}{\Delta t} \tag{34}$$
$$k\Delta t \le t \le (k + 1)\Delta t \qquad 0 \le \epsilon \le \Delta t$$

Combining (33) and (34), the spatial gradient approximation during the time step Δt is given by

$$\left.\frac{\partial\bar{\theta}}{\partial x}\right|_{y=l} = (\theta_{j+1}{}^2 - \theta_{j+1}{}^1 - \theta_j{}^2 + \theta_j{}^1)\epsilon/l\Delta t + (\theta_{j+1}{}^1 - \theta_j{}^1)/l$$
$$\left.\frac{\partial\bar{\theta}}{\partial x}\right|_{y=0} = (\theta_j{}^2 - \theta_j{}^1 - \theta_{j-1}{}^2 + \theta_{j-1}{}^1)\epsilon/l\Delta t + (\theta_j{}^1 - \theta_{j-1}{}^1)/l \tag{35}$$

where superscripts 1 and 2 refer to time steps $k\Delta t$ and $(k + 1)\Delta t$, respectively. Combining (13), (33), and (35) gives

$$\sum_{i=0}^{\infty} \frac{D^{(i)}(y = l, k\Delta t)}{i!} \int_0^{\Delta t} \left\{(\theta_{j+1}{}^2 - \theta_{j+1}{}^1 - \theta_j{}^2 + \theta_j{}^1)\frac{\epsilon^{i+1}}{l\Delta t} + (\theta_{j+1}{}^1 - \theta_j{}^1)\frac{\epsilon^i}{l}\right\} d\epsilon - \sum_{i=0}^{\infty} \frac{D^{(i)}(y = 0, k\Delta t)}{i!} \int_0^{\Delta t} \left\{(\theta_j{}^2 - \theta_j{}^1 - \theta_{j-1}{}^2 + \theta_{j-1}{}^1)\frac{\epsilon^{i+1}}{l\Delta t} + (\theta_j{}^1 - \theta_{j-1}{}^1)\frac{\epsilon^i}{l}\right\} d\epsilon$$

$$= \int_0^l \{\theta\}\Big|_{\Gamma_i} dy \qquad (36)$$

The temporal integration of (36) is evaluated by isolating the time integrable functions as

$$\sum_{i=0}^{\infty} \frac{D^{(i)}(y=l)}{i!}\left[\frac{(\theta_{j+1}{}^2-\theta_j^2)-(\theta_{j+1}{}^1-\theta_j^1)}{l\Delta t}\right]\int_0^{\Delta t}\epsilon^{i+1}\,d\epsilon$$
$$+\sum_{i=0}^{\infty}\frac{D^{(i)}(y=l)}{i!}\left[\frac{\theta_{j+1}{}^1-\theta_j^1}{l}\right]\int_0^{\Delta t}\epsilon^i\,d\epsilon$$
$$-\sum_{i=0}^{\infty}\frac{D^{(i)}(y=0)}{i!}\left[\frac{(\theta_j^2-\theta_{j-1}{}^2)-(\theta_j^1-\theta_{j-1}{}^1)}{l\Delta t}\right]$$
$$\cdot\int_0^{\Delta t}\epsilon^{i+1}\,d\epsilon-\sum_{i=0}^{\infty}\frac{D^{(i)}(y=0)}{i!}\left[\frac{\theta_j^1-\theta_{j-1}{}^1}{l}\right]\int_0^{\Delta t}\epsilon^i\,d\epsilon$$
$$= \int_0^l \{\theta\}\Big|_{\Gamma_i} dy \qquad (37)$$

Rearranging terms, the nodal point water content values can be isolated by

$$[\theta_{j+1}{}^2-\theta_j^2]\sum_{i=0}^{\infty}\frac{D^{(i)}(y=l)}{i!l\Delta t}\int_0^{\Delta t}\epsilon^{i+1}\,d\epsilon$$
$$-[\theta_j^2-\theta_{j-1}{}^2]\sum_{i=0}^{\infty}\frac{D^{(i)}(y=0)}{i!l\Delta t}\int_0^{\Delta t}\epsilon^{i+1}\,d\epsilon$$
$$+[\theta_{j+1}{}^1-\theta_j^1]\sum_{i=0}^{\infty}\frac{D^{(i)}(y=l)}{i!l\Delta t}\int_0^{\Delta t}(\Delta t\epsilon^i-\epsilon^{i+1})\,d\epsilon$$
$$-[\theta_j^1-\theta_{j-1}{}^1]\sum_{i=0}^{\infty}\frac{D^{(i)}(y=0)}{i!l\Delta t}\int_0^{\Delta t}(\Delta t\epsilon^i-\epsilon^{i+1})\,d\epsilon$$
$$= \int_0^l \{\theta\}\Big|_{\Gamma_i} dy \qquad (38)$$

Carrying out the indicated integration in (38) gives

$$[\theta_{j+1}{}^2-\theta_j^2]\sum_{i=0}^{\infty}\frac{D^{(i)}(y=l)}{i!l}\frac{(\Delta t)^{i+1}}{(i+2)}-[\theta_j^2-\theta_{j-1}{}^2]$$
$$\cdot\sum_{i=0}^{\infty}\frac{D^{(i)}(y=0)}{i!l}\frac{(\Delta t)^{i+1}}{(i+2)}+[\theta_{j+1}{}^1-\theta_j^1]\sum_{i=0}^{\infty}\frac{D^{(i)}(y=l)}{i!l}$$
$$\cdot\frac{(\Delta t)^{i+1}}{(i+1)(i+2)}-[\theta_j^1-\theta_{j-1}{}^1]\sum_{i=0}^{\infty}\frac{D^{(i)}(y=0)}{i!l}\frac{(\Delta t)^{i+1}}{(i+1)(i+2)}$$
$$= \int_0^l \{\theta\}\Big|_{\Gamma_i} dy \qquad (39)$$

In a simplified notation, (39) can be rewritten as

$$\bar{\bar{D}}_l[\theta_{j+1}{}^2-\theta_j^2]-\bar{\bar{D}}_0[\theta_j^2-\theta_{j-1}{}^2]=$$
$$-\bar{D}_l[\theta_{j+1}{}^1-\theta_j^1]+\bar{D}_0[\theta_j^1-\theta_{j-1}{}^1]+\int_0^l\{\theta\}\Big|_{\Gamma_i}dy \qquad (40)$$

where

$$\bar{\bar{D}}_\xi=\sum_{i=0}^{\infty}\frac{D^{(i)}(y=\xi)}{i!(i+2)}\frac{(\Delta t)^{i+1}}{l} \qquad \xi=0,l$$
$$\bar{D}_\xi=\sum_{i=0}^{\infty}\frac{D^{(i)}(y=\xi)(\Delta t)^{i+1}}{(i+2)!l} \qquad \xi=0,l \qquad (41)$$

For the Δt duration space-time surface assumed linear with respect to time the temporal differentials of soil water diffusivity in (41) are given by

$$D^{(N)}\equiv\frac{\partial^N D}{\partial t^N}=\frac{\partial^N D}{\partial\theta^N}\left(\frac{\partial\theta}{\partial t}\right)^N \qquad (42)$$

where N denotes the order of the differential operator.

MODEL APPLICATIONS (ALTERNATION THEOREM)

The normalized moisture transport problem for constant diffusivity [*Hromadka and Guymon*, 1980*a*, *b*] is given by

$$\frac{\partial^2\theta}{\partial x^2}=\frac{\partial\theta}{\partial t} \qquad x\in\Omega \qquad (43)$$

where

$$\Omega=\{x|0\le x\le 1\}$$
$$\theta\,(x,t=0)=1 \qquad x\in\Omega \qquad (44)$$
$$\theta\,(x=0,1;t>0)=0$$

The problem domain Ω is assumed discretized into two finite elements (Ω_1, Ω_2) of equal length by three nodal values (θ_1, θ_2, θ_3) where (x_1, x_2, x_3) = (0, 0.5, 1.). Because of the boundary conditions of (44) the resulting system of modeled linear equations reduces to a single equation of one unknown, θ_2. In order to evaluate the effectiveness of using the alternation theorem approach to modeling (43) and (44) the finite element, finite difference, and nodal domain integration solutions will also be determined for comparison purposes.

The Galerkin version of the weighted residual process can be used to approximate (1) and (43) by the finite element method. The solution domain is discretized into the union of $(n-1)$ finite elements (21) by

$$\Omega=\bigcup_{i=1}^{n-1}\Omega_i \qquad (45)$$

The water content is utilized as the state variable and is approximated within each finite element by

$$\theta(x,t)=\sum_{j=1}^{n}N_j(x)\,\theta_j(t) \qquad (46)$$

where N_j is the appropriate linearly independent shape functions and θ_j is the state variable values at element-nodal points designated by the general summation index j.

The Galerkin technique utilizes the set of shape functions as the weighting functions, which indicates that the corresponding finite element representation for the infiltration process is

$$\int\left\{\frac{\partial}{\partial x}\left[D(\theta)\frac{\partial\theta}{\partial x}\right]-\frac{\partial\theta}{\partial t}\right\}N_j\,dx=0 \qquad (47)$$

Integration by parts expands (47) into the form

$$\sum_{i=1}^{n}\left\{D(\theta)\frac{\partial\theta}{\partial x}N_j\Big|_{S_i}-\int_{\Omega_i}\left[D(\theta)\frac{\partial\theta}{\partial x}\frac{\partial N_j}{\partial x}+N_j\frac{\partial\theta}{\partial t}\right]dx\right\}=0 \qquad (48)$$

where S_i represents the external end points of the one-dimensional finite element Ω_i. The first term within the braces sums to zero for interior elements and also satisfies the usual specified (flux) boundary conditions of the problem for exterior finite elements. The remaining integral term is solved by substituting the appropriate element approximations and shape functions into the integrand and solving by numerical integration. A convenient approach for dealing with the nonlinearity of (48) is to assume the diffusivity function to be constant within each finite element during a finite time interval Δt in order to carry out the integration [*Guymon and Luthin,* 1974]. *Hromadka and Guymon* [1980*a*] examined some approaches in determining appropriate values of diffusivity for use in this method of linearizing. The Crank-Nicolson time advancement approximation has been widely used [*Hayhoe,* 1974; *Desai,* 1979] to perform the time integration of (1) and (43).

The Crank-Nicolson formulation reduces (48), where values of soil water diffusivity are assumed constant within each finite element, into a system of linear equations expressed in matrix form as

$$\left\{\mathbf{P} + \frac{\Delta t}{2}\mathbf{S}\right\}\theta^{k+1} = \left\{\mathbf{P} - \frac{\Delta t}{2}\mathbf{S}\right\}\theta^{k} \qquad (49)$$

where $\mathbf{P}$ is a symmetrical capacitance matrix and is a function of element nodal global coordinates, $\mathbf{S}$ is a symmetrical stiffness matrix and is a function of element nodal global coordinates and constant finite element diffusivity coefficients (during timestep Δt), Δt is the finite time step increment, and θ^k is the vector of nodal state variable approximations (volumetric water content) at time step k.

For a linear polynomial trial function the element matrices determined from (48) are given by

$$\mathbf{S}\begin{Bmatrix}\theta_i\\ \theta_j\end{Bmatrix} + \hat{\mathbf{P}}\begin{Bmatrix}\dot\theta_i\\ \dot\theta_j\end{Bmatrix} = \frac{D_i}{l_i'}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\theta_i\\ \theta_j\end{Bmatrix} + \frac{l_i'}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}\dot\theta_i\\ \dot\theta_j\end{Bmatrix} \qquad (50)$$

where D_i is the quasi-constant diffusivity within element i, $\mathbf{S}$ and $\hat{\mathbf{P}}$ are element stiffness and capacitance matrices, respectively, and (θ_i, θ_j) and $(\dot\theta_i, \dot\theta_j)$ refer to the element nodal and time derivative of nodal moisture content values, respectively, for an element of length l_i'.

For the linear temporal trial function the nodal domain integration approximation of (40) and (41) can be written analogously to (49) and (50) as

$$\{\hat{\mathbf{P}} + \bar{\mathbf{S}}\}\ \theta^{k+1} = \{\hat{\mathbf{P}} - \hat{\mathbf{S}}\}\ \theta^{k} \qquad (51)$$

where the element matrices composing the global $\mathbf{S}$ matrices of (51) are given by

$$\bar{\mathbf{S}} = \bar{D}_\xi\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \qquad \hat{\mathbf{S}} = \hat{D}_\xi\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \qquad (52)$$

and for $\alpha \to \infty$

$$\hat{\mathbf{P}} = \frac{l}{8}\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix} \qquad (53)$$

Hromadka and Guymon [1980*a*] rewrite the capacitance matrix $\hat{\mathbf{P}}$ in (50) as

$$\hat{\mathbf{P}}(\eta) = \frac{l}{2(\eta+1)}\begin{bmatrix}\eta & 1\\ 1 & \eta\end{bmatrix} \qquad (54)$$

where the Galerkin approximation in (49) leads to

$$\hat{\mathbf{P}}(2) = \frac{l}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} \qquad (55)$$

The nodal integration approximation with $\alpha \to \infty$ leads to a type of subdomain approximation

$$\hat{\mathbf{P}}(3) = \frac{l}{8}\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix} \qquad (56)$$

The finite difference approach is given by

$$\lim_{\eta\to\infty}\hat{\mathbf{P}}(\eta) = \frac{l}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad (57)$$

Application of the alternation theorem to the sinusoidal estimate of (20) for θ in the normalized problem of (43) is made for $\alpha = 1$, where

$$\theta = \theta_2 \sin \pi x \qquad 0 \le x \le 1 \qquad (58)$$

The best linear approximation $\bar\theta$ for θ in subdomain $\hat\Omega_1$ is found from (32). For $0 < \mu < 0.5$ and $x \in \hat\Omega_1$, solution of (29) gives (for $\alpha = 1$)

$$\mu = \frac{1}{\pi}\cos^{-1}\left[\frac{2}{\pi}\right] \simeq 0.28 \qquad (59)$$

Thus the maximum error e in the linear approximation $\bar\theta$ for θ on $\hat\Omega_1$ occurs at $x = (0, \mu, \tfrac{1}{2})$. This error is evaluated from (23*b*) as

$$e = \theta_2\left(\tfrac{1}{2}\sin \pi\mu - \mu\right) \simeq 0.105\theta_2 \qquad (60)$$

Thus the best linear approximation $\bar\theta$ for θ on $(\hat\Omega_1, \hat\Omega_2)$ is

$$\begin{aligned}\bar\theta(x) &= 2\theta_2 x + 0.105\theta_2 \qquad x \in \hat\Omega_1\\ \bar\theta(x) &= 2\theta_2(1-x) + 0.105\theta_2 \qquad x \in \hat\Omega_2\end{aligned} \qquad (61)$$

Selection of other values of α in (20) would result in a different linear approximation $\bar\theta$ in (61).

From (19), solution of (43) and (44) gives

$$\left.\frac{\partial\bar\theta}{\partial x}\right|_{\Gamma_j} = -4\theta_2 \qquad (62)$$

$$\frac{\partial}{\partial t}\int_{\Omega_2}\bar\theta\ dx = \left(\frac{e}{2} + \frac{3}{8}\right)\frac{\partial\theta_2}{\partial t} \simeq 0.4275\ \frac{\partial\theta_2}{\partial t} \qquad (63)$$

where the e term in (63) serves as a type of weighting factor to the θ_2 nodal point approximation in the nodal integration formulation of (19).

For the study problem of (43) and (44) the presented domain numerical solutions result in the expression [*Hromadka and Guymon,* 1980*a, b*]

$$\lim_{\Delta t\to 0}\theta_2(t) = \exp\left[-8\left(\frac{\eta+1}{\eta}\right)t\right] \qquad t \ge 0 \qquad (64)$$

where η is the entry in the capacitance matrix $\mathbf{P}(\eta)$ of (54). Solution of (62) and (63) gives

$$-4\theta_2(t) = 0.4275\ \frac{\partial\theta_2}{\partial t} \qquad (65)$$

therefore

$$\theta_2(t) = \exp\left[(-9.357)t\right] \qquad t \ge 0 \qquad (66)$$

TABLE 1. Domain Solution η Values for Test Problem

Approximation	Equation	Equivalent η*
Nodal integration–linear shape function	(56)	3
Galerkin-linear shape function	(55)	2
Finite difference	(57)	∞
Linear approximation to parabola	(72)	7
Linear approximation to sinusoidal estimate	(66)	5.9

Reference text equations (43) and (44).
*Reference text equation (54).

As a second selection for the $\hat{\theta}$ approximation, a second-order polynomial for θ on Ω in the solution of (43) and (44) is

$$\hat{\theta}(x) = 4\theta_2(x - x^2) \qquad x \in \Omega \tag{67}$$

Analogous to the sinusoidal approximation, the best linear approximation for the parabola $\hat{\theta}$ function on Ω_1 is

$$\bar{\theta}(x) = 2\theta_2 + e \qquad x \in \Omega_1 \tag{68}$$

where μ and e are

$$\mu = \tfrac{1}{4} \qquad e = \theta_2/8 \tag{69}$$

The maximum θ error to the parabola $\hat{\theta}$ in Ω_1 occurs at $x = (0, \tfrac{1}{4}, \tfrac{1}{2})$. Therefore

$$\bar{\theta}(x) = 2\theta_2 x + \frac{\theta_2}{8} \qquad x \in \Omega_1$$
$$\bar{\theta}(x) = 2\theta_2(1 - x) + \frac{\theta_2}{8} \qquad x \in \Omega_2 \tag{70}$$

Combination of (6) and (70) gives

$$-4\theta_2(t) = \frac{7}{16}\frac{\partial\theta_2}{\partial t} \tag{71}$$

$$\lim_{\Delta t \to 0} \theta_2(t) = \exp\left[\left(-\frac{64}{7}\right)t\right] \qquad t \geq 0 \tag{72}$$

Table 1 summarizes computed or equivalent values of η corresponding to (64) for the various domain approximations of (43) and (44). Table 2 gives values of the tested domain solutions for comparison to the analytical solution of the example problem at $x = 0.50$.

From Table 2 it can be seen that a numerical model using the alternation theorem to approximate a higher-order trial function approximation $\hat{\theta}$ for θ increases numerical model accuracy (for the problem studied) in comparison to the standard Galerkin finite element and linear nodal domain integration approaches. The finite difference numerical approximation, however, gives the best numerical estimates for θ during the initial test problem solution. After normalized time $t = 0.12$, however, the finite difference approximation increasingly overestimates the analytic solution for θ.

The above results suggest that the parameter η of the element capacitance matrix in (54) should vary as a function of time in order to obtain a more accurate numerical approximation. The following section develops such a numerical model which determines η as a function of time for each finite element.

Approximation Improvement by Use of Adjusted Linear Model (Nodal Domain Integration Method)

In this section a second method of modeling a higher-order or more complex family of trial functions by a linear trial function approximation is presented. For the one-dimensional soil water transport problem studied, this approach assumes that the matrix diagonal entry η (54) is a function of time and that the spatial integration and gradient evaluation of a higher-order approximation $\hat{\theta}$ of θ can be equated on Ω_j to an approximation based on an adjusted linear trial function system $\bar{\theta}$.

Let $\bar{\theta}$ be an approximation function of a higher-order approximation $\hat{\theta}$ of θ, where the spatial gradients of $\bar{\theta}$ on Γ_j are defined by

$$\left\{\frac{\partial\bar{\theta}}{\partial x}\right\}\Bigg|_{\Gamma_j} \equiv \frac{(\theta_{j+1} - \theta_j)}{l_j'} - \frac{(\theta_j - \theta_{j-1})}{l_{j-1}'} \tag{73}$$

where l_j' is the length of finite element j.

A spatial gradient adjustment function $c(x, t)$ is defined by

$$c(x, t) = (\partial\hat{\theta}/\partial x)/(\partial\bar{\theta}/\partial x) \qquad 0 < c < \infty$$
$$c(x, t) = 1 \qquad \text{otherwise} \tag{74}$$

Therefore it is assumed that

$$D\frac{\partial\hat{\theta}}{\partial x} = Dc\frac{\partial\bar{\theta}}{\partial x} \tag{75}$$

where

$$\left\{D\frac{\partial\hat{\theta}}{\partial x}\right\}\Bigg|_{\Gamma_j} = \left\{Dc\frac{\partial\bar{\theta}}{\partial x}\right\}\Bigg|_{\Gamma_j} \tag{76}$$

TABLE 2. Numerical Solution of Normalized Soil Moisture Transport Problem

Time	$\eta = 2$	$\eta = 3$	$\eta = 5.9$	$\eta = 7$	$\eta = \infty$	Exact
0.01	0.887	0.889	0.911	0.913	0.923	0.999
0.02	0.787	0.808	0.829	0.833	0.852	0.975
0.03	0.698	0.726	0.755	0.760	0.787	0.918
0.04	0.619	0.653	0.688	0.694	0.726	0.846
0.05	0.549	0.587	0.626	0.633	0.670	0.772
0.10	0.301	0.344	0.392	0.401	0.449	0.474
0.15	0.165	0.202	0.246	0.254	0.301	0.290
0.20	0.091	0.118	0.154	0.161	0.202	0.177
0.25	0.050	0.069	0.096	0.102	0.135	0.108
0.30	0.027	0.041	0.060	0.064	0.091	0.066

One variable nodal point.

On Γ_j, define

$$Dc = A(t) \qquad k\Delta t \le t \le (k+1)\Delta t \tag{77}$$

such that

$$A(k\Delta t + \epsilon) = \sum_{i=0}^{\infty} A^{(i)}(k\Delta t)\frac{\epsilon^i}{i!} \qquad 0 \le \epsilon \le \Delta t \tag{78}$$

where (i) represents the ith order temporal partial differential operator. Then

$$\left\{D\frac{\partial\theta}{\partial x}\right\}\Bigg|_{\Gamma_j} = \left\{\sum_{i=0}^{\infty} A^{(i)}(k\Delta t)\frac{\epsilon^i}{i!}\frac{\partial\bar{\theta}}{\partial x}\right\}\Bigg|_{\Gamma_j} \tag{79}$$

A function $\eta(t)$ is defined by

$$\int_{\Omega_j}\bar{\theta}\,dx = \frac{l_j}{2[\eta(t)+1]}[\theta_{j-1} + 2\theta_j\eta(t) + \theta_{j+1}] \tag{80}$$

where for modeling purposes $\eta(t)$ is restricted to values

$$\eta(t) \ge 2 \tag{81}$$

The value of 3 in (81) corresponds to a first-order polynomial subdomain approximation for θ, whereas $\eta(t) = 2$ corresponds to a finite element (Galerkin) approach, and $\eta(t) \to \infty$ corresponds to a finite difference approximation.

The $\bar{\theta}$ approximator is also defined to have the property

$$\int_{\Omega_j}\theta\,dx = \int_{\Omega_j}\bar{\theta}\,dx \qquad \eta(t) \ge 2 \tag{82a}$$

as defined by (80).

$$\int_{\Omega_j}\theta\,dx = \frac{l_j}{8}[\theta_{j-1} + 6\theta_j + \theta_{j+1}] \qquad \eta(t) < 2 \tag{82b}$$

Substituting (79) and (82) into (36) gives the nodal domain integration statement

$$\int_{\Delta t}\left\{\sum_{i=0}^{\infty} A^{(i)}(k\Delta t)\frac{\epsilon^i}{i!}\frac{\partial\bar{\theta}}{\partial x}\right\}\Bigg|_{\Gamma_j} = \frac{l_j[\theta_{j-1} + 2\theta_j\eta(k\Delta t + \Delta t) + \theta_{j+1}]}{2[\eta(k\Delta t + \Delta t) + 1]} - \frac{l_j[\theta_{j-1} + 2\theta_j\eta(k\Delta t) + \theta_{j+1}]}{2[\eta(k\Delta t) + 1]} \tag{83}$$

where

$$\eta(k\Delta t + \epsilon) = \sum_{i=0}^{\infty}\eta^{(i)}(k\Delta t)\frac{\epsilon^i}{i!} \qquad 0 \le \epsilon \le \Delta t \tag{84}$$

Analogous to the development leading to (41),

$$\bar{A}(\xi) = \frac{1}{l_j}\sum_{i=0}^{\infty} A_i\frac{(\xi)(\Delta t)^{i+1}}{i!(i+2)} \qquad \xi = (0, l)$$
$$\tilde{A}(\xi) = \frac{1}{l_j}\sum_{i=0}^{\infty} A_i\frac{(\xi)(\Delta t)^{i+1}}{(i+2)!} \qquad \xi = (0, l) \tag{85}$$

where for modeling purposes it is assumed that second-order (and higher) temporal differentials are negligible and

$$A_0 = [cD]|_{k\Delta t}$$
$$A_1 = \left[c\frac{\partial D}{\partial\theta}\frac{\partial\theta}{\partial t} + D\frac{\partial c}{\partial t}\right]\Bigg|_{k\Delta t} \tag{86}$$
$$A_2 = \left[c\frac{\partial^2 D}{\partial\theta^2}\left(\frac{\partial\theta}{\partial t}\right)^2 + 2\frac{\partial c}{\partial t}\frac{\partial D}{\partial\theta}\frac{\partial\theta}{\partial t}\right]\Bigg|_{k\Delta t}$$

and for small Δt, $(\partial^2\theta/\partial t^2) = (\partial c^2/\partial t^2) = 0$. Thus analogous to (51), (52), and (53),

$$\bar{\mathbf{s}} = \bar{A}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$
$$\tilde{\mathbf{s}} = \tilde{A}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$
$$\bar{\mathbf{p}} = \frac{l_j}{2(\bar{\eta}+1)}\begin{bmatrix}\eta & 1\\ 1 & \bar{\eta}\end{bmatrix} \tag{87}$$
$$\tilde{\mathbf{p}} = \frac{l_j}{2(\tilde{\eta}+1)}\begin{bmatrix}\tilde{\eta} & 1\\ 1 & \tilde{\eta}\end{bmatrix}$$

where $\bar{\eta} = \eta(k\Delta t + \Delta t)$ and $\tilde{\eta} = \eta(k\Delta t)$.

From the above the soil water transport problem may be modeled by an appropriately defined linear trial function approximation set which incorporates some of the benefits of a higher-order family of approximations. Thus additional numerical accuracy may be achieved while retaining the symmetric matrix formulation characteristic of a linear polynomial approximation of θ.

MODEL APPLICATIONS (LINEAR APPROXIMATION ADJUSTMENT)

The normalized transport problem of (43) and (44) is reanalyzed using a five nodal point discretization of Ω with $(x_1, x_2, x_3, x_4, x_5) = (0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1)$. A Galerkin finite element or finite difference numerical approximation model for this problem follows from the preceding sections.

For θ assumed to be described by a second-order polynomial such that

$$\theta = N_1\theta_{j-1} + N_2\theta_j + N_3\theta_{j+1} \qquad x \in \Omega_j \tag{88}$$

then from (74) and (82)

$$c(t) = 1$$
$$\eta(t) = 11 \tag{89}$$

As another example of a higher-order approximation $\bar{\theta}$ of θ on Ω, a fourth-order polynomial approximation of θ is given by

$$\theta = \sum_{i=1}^{5} N_i\theta_i \qquad x \in \Omega \tag{90}$$

For the fourth-order approximation, (87) is determined for each Δt time step by solving (74), (80), and (82). For a normalized time step of $\Delta t = 0.01$, $n(t)$ and $c(t)$ were modeled as a constant during each time step, ignoring the time variation of both adjustment terms.

Computer simulation results for numerical models based on the Galerkin finite element method ($\eta = 2$), linear subdomain method ($\eta = 3$), finite difference method ($\eta = \infty$), nodal domain integration using (89) ($\eta = 11$), nodal domain integration using an adjusted linear approximation of a fourth-order polynomial approximation, and a fourth-order polynomial subdomain approximation for the test problem are given in Table 3. From Table 3 the true fourth-order subdomain approximation gave the most accurate results, but the

TABLE 3. Numerical Solution of Normalized Moisture Transport Problem

Time	η = 2*		η = 3†		η = 11‡		η = ∞§		Fourth Order†		Adjusted Linear‡		Analytic	
	x = 0.25	x = 0.5	x = 0.25	x = 0.5	x = 0.25	x = 0.5	x = 0.25	x = 0.5	x = 0.25	x = 0.5	x = 0.25	x = 0.5	x = 0.25	x = 0.5
0.01	0.802	1.041	0.823	1.017	0.851	0.989	0.861	0.981	0.864	1.003	0.864	1.013	0.923	0.999
0.02	0.701	0.970	0.716	0.960	0.743	0.941	0.755	0.933	0.760	0.961	0.761	0.967	0.789	0.975
0.03	0.627	0.881	0.637	0.882	0.660	0.876	0.671	0.873	0.676	0.898	0.678	0.902	0.690	0.918
0.04	0.564	0.796	0.572	0.802	0.592	0.807	0.602	0.807	0.606	0.828	0.608	0.830	0.615	0.846
0.05	0.508	0.718	0.515	0.727	0.533	0.739	0.543	0.743	0.546	0.758	0.548	0.759	0.553	0.772
0.10	0.302	0.427	0.310	0.439	0.327	0.461	0.335	0.472	0.331	0.469	0.331	0.468	0.336	0.474
0.15	0.179	0.254	0.187	0.264	0.202	0.285	0.209	0.295	0.202	0.287	0.201	0.285	0.205	0.290
0.20	0.107	0.151	0.113	0.159	0.125	0.176	0.131	0.185	0.123	0.175	0.123	0.174	0.125	0.177
0.25	0.063	0.090	0.068	0.096	0.077	0.109	0.082	0.116	0.075	0.107	0.075	0.106	0.076	0.018
0.30	0.038	0.053	0.041	0.058	0.048	0.067	0.051	0.072	0.046	0.065	0.046	0.064	0.047	0.066

Three variable nodal points.
*Galerkin finite element analog.
†Subdomain approximation.
‡Nodal domain integration method.
§Finite difference method.

nodal domain integration model closely matched these results. Thus the numerical accuracy produced by a fourth-order approximation is closely matched by a first-order approximation, significantly reducing computer memory requirements. Additionally, the computer computation requirements in solving (74) and (82) are offset by the reduction in a higher-order approximation matrix computational effort. Values of $\eta(t)$ and $c(t)$ computed for the (fourth order) linear adjusted model were approximately 10.5 and 1.0, respectively. This may explain the good results obtained by the linear adjusted model using (88) and (89).

APPLICATION OF LINEAR APPROXIMATION ADJUSTMENT APPROACH TO A NONLINEAR PROBLEM

The numerical model given by (83), (87), and (89) was applied to a sharp wetting front problem of soil water infiltration into an air dry horizontal column [*Hayhoe*, 1978; *Hromadka and Guymon*, 1980*c*]. The analytical value of soil water diffusivity for Hanford sandy loam [*Reichardt et al.*, 1972] was selected in order to provide a sharp wetting front through the soil column, causing the numerical analysis of moisture flow in the soil to be difficult. The quasi-analytic solution advanced by *Philip and Knight* [1974] and utilized by *Hayhoe* [1978] was used for this study.

Equation (1) was solved subject to the initial condition

$$\theta(x, t) = 0 \qquad t = 0 \qquad 0 \le x \le L \tag{91}$$

and the boundary conditions

$$\theta(0, t) = 1 \qquad \theta(L, t) = 0 \qquad t > 0 \tag{92}$$

where the soil water diffusivity (cm^2 min^{-1}) is given by

$$D(\theta) = 0.9 \times 10^{-3} \exp(8.36\theta) \qquad \theta > 0$$
$$D(\theta) = 0.9 \times 10^{-3} \qquad \theta = 0 \tag{93}$$

and θ is the volumetric water content.

Because of the nature of the soil water diffusivity function of (93) the temporal variation of diffusivity is extremely important during the Δt time step. Table 4 contains various values of time step magnitudes Δt (in minutes) and time series expansion terms in the numerical model of (1) by (83) at time $t = 16.5$ min. For the numerical model a spatial discretization of 0.5 cm was used where the total column length was set at 5.0 cm to correspond to the model results of *Hayhoe* [1978].

CONCLUSIONS

Two techniques of modeling a higher-order trial function approximation of soil moisture transport using an improved linear trial function approximation set have been developed. Both techniques retain the smaller symmetrical matrix systems associated with numerical models of soil moisture transport based on a linear polynomial trial function but increase the numerical accuracy of the model by incorporating some of the benefits of a higher-order approximation.

Because the various numerical methods considered (finite difference, Galerkin finite element, subdomain method) are available in the proposed model, it is concluded that the proposed numerical approach may lead to a generalized modeling method for all soil moisture transport problems. The computer code used for each simulation is identical except for a variation in the capacitance matrix entry η. Therefore a comparison of numerical efficiency between the finite differ-

TABLE 4. Comparison of Numerical Model Results at Time t = 16.5 Minutes

x, cm	Analytic*	Δt = 0.1 min, i = 0	Δt = 0.1 min, i = 2	Δt = 0.1 min, i = 5	Δt = 0.3 min, i = 0	Δt = 0.3 min, i = 2	Δt = 0.3 min, i = 5
0.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
0.5	0.99	0.99	0.99	0.99	0.98	0.99	0.99
1.0	0.97	0.97	0.97	0.97	0.97	0.97	0.97
1.5	0.95	0.95	0.95	0.95	0.94	0.95	0.95
2.0	0.92	0.93	0.92	0.92	0.92	0.93	0.93
2.5	0.88	0.90	0.89	0.89	0.88	0.90	0.90
3.0	0.84	0.87	0.85	0.85	0.84	0.87	0.86
3.5	0.78	0.82	0.78.	0.78	0.82	0.82	0.82
4.0	0.67	0.39	0.63	0.64	0.05	0.37	0.40
4.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Values of water content. The i is the number of temporal Taylor series terms (diffusivity function) included, and Δt is the time step magnitude.

*Results from *Hayhoe* [1978].

ence, Galerkin finite element, subdomain method, and the proposed nodal domain integration approach is provided.

NOTATION

- A flux adjustment factor.
- α sinusoidal curve trial function adjustment factor.
- D soil water diffusivity.
- (λ, β) coefficients of $\bar{\theta}$ linear function approximation for $\hat{\theta}$.
- c gradient adjustment factor.
- ϵ local time coordinate $0 \leq \epsilon \leq \Delta t$.
- e alternation theorem error of $\bar{\theta}$ approximation for $\hat{\theta}$.
- μ point of relative maximum error of $\bar{\theta}$ approximation for $\hat{\theta}$ in Ω_r.
- (i) partial differential operator (order).
- k time step increment number.
- L length of one-dimensional domain.
- l_j length of nodal domain j.
- l' length of finite element spatial domain.
- l length of nodal domain for constant element discretization.
- M_m temporal shape function.
- N_r spatial shape function.
- n number of nodal points in Ω.
- $\eta(t)$ integration adjustment factor as a function of time.
- θ unsaturated volumetric water content.
- θ_j value of θ at node j.
- $\hat{\theta}$ trial function approximation for θ.
- θ_r^m $\hat{\theta}(x = x_r, t = m\Delta t)$.
- $\bar{\theta}$ linear polynomial approximation for $\hat{\theta}$.
- η element capacitance matrix diagonal entry.
- x_j spatial coordinate of node j.
- Δt time step (constant).
- t time.
- Γ_t limits of time step integration.
- Z local spatial coordinate in finite element spatial domain.
- Ω domain of problem definition.
- Ω_j nodal domain j.
- $\hat{\Omega}_j$ finite element domain j.
- Γ_j boundary of Ω_j.
- $\mathbf{P}$ global capacitance matrix.
- $\hat{\mathbf{P}}$ finite element capacitance matrix.
- $\bar{\mathbf{P}}$ nodal domain capacitance matrix.
- $\hat{\mathbf{P}}(\eta)$ finite element capacitance matrix as a function of η.
- $\mathbf{S}$ global stiffness matrix.
- $\hat{\mathbf{S}}$ finite element stiffness matrix.
- $\tilde{\mathbf{S}}, \bar{\mathbf{S}}$ nodal domain stiffness matrices.

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