Subdomain Integration Model of Ground-Water Flow

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Introduction

Numerical solution of the governing differential equations describing one-dimensional ground-water flow has received substantial attention since the advent of modern computers. Generally, numerical models are developed which employ the well-known finite-difference or Galerkin finite element methods (1,2) to approximate the governing equations, resulting in a model which can be solved generally only with computers of at least the minicomputer class. Recently, the "method-of-lines" (3) was used to solve the nonlinear unconfined ground-water flow equations resulting in a numerical model which can be accommodated by a programmable hand-held calculator. The algorithm used a "shooting method" which required an iteration process to obtain the desired accuracy. The main purpose of this paper is to present another approach to solving nonlinear (and linear) problems such as ground-water flow processes which also may be accommodated by programmable calculators. The numerical approach used is the subdomain integration version of the weighted residual methods as applied to solving for spatial coordinates as a function of the ground-water (or piezometric) surface.

The objectives of this paper are threefold. The first objectives is to present the subdomain integration numerical method as applied to a specific class of one-dimensional transport problems. Hromadka and Guymon (6,7,8) developed this numerical modeling procedure in detail and compare modeling efficiency to the well known finite difference and Galerkin finite element methods, and conclude that the subdomain integration procedure leads to a more accurate numerical model for the various problems tested. Extension of the modeling method to one-dimensional and two-dimensional linear and nonlinear advection-diffusion problems are the subject of current papers (9,10,11).

The second objective of this paper is to determine a subdomain integration

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numerical model which solves for spatial coordinates rather than solving for the governing flow equation's state variable. This approach somewhat eliminates nonlinearity (due to state variable dependent parameters) because the nonlinear parameters are evaluated at a constant value of the state variable along the boundaries of each subdomain.

The third objective is to simplify the resulting subdomain integration numerical model into an approximation which can be accommodated by programmable calculators. For the specialized problems tested, the simplified subdomain integration approximation produced good results when compared to available analytic solutions.

Governing One-Dimensional Ground-Water Flow Equations

One dimensional, unsteady ground-water flow in a confined homogeneous aquifer of a nearly uniform thickness is generally described by a linear partial differential equation of the form

$$\alpha \frac{\partial^2 h}{\partial x^2} = \frac{\partial h}{\partial t} \quad (1)$$

subject to appropriate boundary and initial conditions. In Eq. 1, h = a convenient reference of piezometric head (Fig. 3); x and t = spatial and temporal coordinates; and α = the quotient of transmissivity T, and effective porosity, n. Examples of approximately one-dimensional ground-water flow include the movement of water between a stream and the aquifer in response to a change in stage, and aquifer recharge from streams, canals, and irrigation ditches due to a sudden increase in stage.

The nonlinear partial differential equation describing one-dimensional unconfined ground-water flow is the well known Boussinesq equation

$$\frac{\partial}{\partial x}\left(K_s h \frac{\partial h}{\partial x}\right) = n \frac{\partial h}{\partial t} \quad (2)$$

in which K_s = the saturated hydraulic conductivity; and h = the hydraulic head. Due to the nonlinearity of Eq. 2, only a few quasianalytical solutions exist for select problems (4,5).

Numerical Model

The subdomain integration method (6–10) is applied to the governing flow equation of unconfined (and confined) ground-water flow. The boundary and initial conditions are assumed defined such that the spatial coordinate x can be described as a function of h (Fig. 1). The h-axis domain Ω is discretized by n nodal points into n subdomains Ω_j such that

$$\Omega = \bigcup_{j=1}^{n} \Omega_j \quad (3)$$

in which $$\Omega_j \cap \Omega_{j+1} = \frac{1}{2}(h_j + h_{j+1}); \quad j = 1, 2, \ldots, n-1 \quad (4)$$

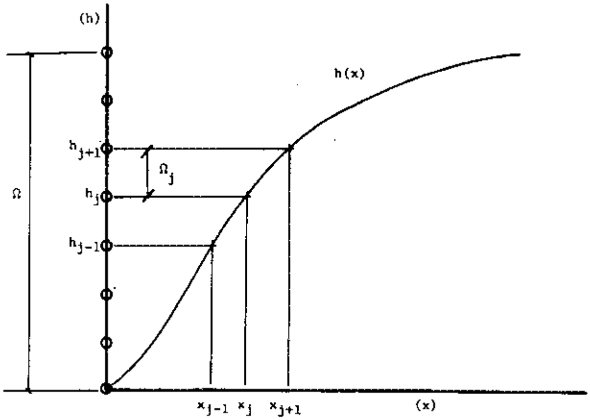

FIG. 1.—Nodal Point Distribution on h-Axis Global Domain Ω

For an interior subdomain Ω_j, $1 < j < n$, let

$$h' = \Omega_{j-1} \cap \Omega_j = \frac{1}{2}(h_{j-1} + h_j); \quad h'' = \Omega_j \cap \Omega_{j+1} = \frac{1}{2}(h_j + h_{j+1}) \quad \ldots\ldots (5)$$

$$x' = x(h'); \quad x'' = x(h'') \quad \ldots\ldots (6)$$

The integration of the nonlinear Eq. 2 over Ω_j is modeled by

$$\left(K_s h \frac{dh}{dx}\right)\bigg|_{x''} - \left(K_s h \frac{dh}{dx}\right)\bigg|_{x'} = n \frac{d}{dt}\int_{x'}^{x''} h\, dx + nh' \frac{dx'}{dt} - nh'' \frac{dx''}{dt} \quad \ldots (7)$$

For a linear trial function approximation on subdomain Ω_j (Fig. 2)

$$\int_{x'}^{x''} h\, dx = \frac{1}{8}\left[(x_j - x_{j-1})(h_{j-1} + 3h_j) + (x_{j+1} - x_j)(h_{j+1} + 3h_j)\right] \quad \ldots\ldots (8)$$

in which each h_i is constant and $x_i = x_i(t)$. Thus

$$\frac{d}{dt}\int_{x'}^{x''} h\, dx = \frac{1}{8}\left[(h_{j-1} + 3h_j)\left(\frac{dx_j}{dt} - \frac{dx_{j-1}}{dt}\right)\right.$$

$$\left. + (h_{j+1} + 3h_j)\left(\frac{dx_{j+1}}{dt} - \frac{dx_j}{dt}\right)\right] \quad \ldots\ldots (9)$$

$$\frac{dx'}{dt} = \frac{1}{2}\left(\frac{dx_{j-1}}{dt} + \frac{dx_j}{dt}\right); \quad \frac{dx''}{dt} = \frac{1}{2}\left(\frac{dx_j}{dt} + \frac{dx_{j+1}}{dt}\right) \quad \ldots\ldots (10)$$

Equation 9 can be rewritten as

$$\frac{d}{dt}\int_{x'}^{x''} h\, dx = \frac{dx_{j-1}}{dt}\left(-\frac{h_{j-1} + 3h_j}{8}\right) + \frac{dx_j}{dt}\left(\frac{h_{j-1} - h_{j+1}}{8}\right)$$

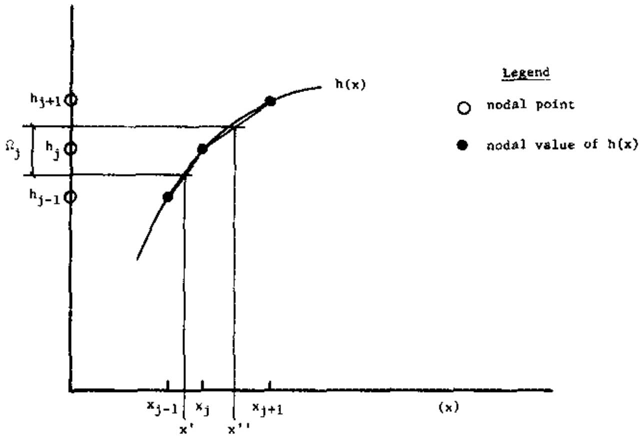

FIG. 2.—Linear Trial Function Approximation on Subdomain Ω_j

$$+ \frac{dx_{j+1}}{dt}\left(\frac{h_{j+1}+3h_j}{8}\right) \quad (11)$$

and $$h'\frac{dx'}{dt} - h''\frac{dx''}{dt} = \frac{dx_{j-1}}{dt}\left(\frac{h_{j-1}+h_j}{4}\right) + \frac{dx_j}{dt}\left(\frac{h_{j-1}-h_{j+1}}{4}\right)$$

$$- \frac{dx_{j+1}}{dt}\left(\frac{h_j+h_{j+1}}{4}\right) \quad (12)$$

From the preceding linear trial function approximations, Eq. 7 is rewritten as

$$\frac{d}{dt}\int_{x'}^{x''} h\,dx + h'\frac{dx'}{dt} - h''\frac{dx''}{dt} = \frac{dx_{j-1}}{dt}\left(\frac{h_{j-1}-h_j}{8}\right)$$

$$+ \frac{dx_j}{dt}\left(\frac{3h_{j-1}-3h_{j+1}}{8}\right) + \frac{dx_{j+1}}{dt}\left(\frac{h_j-h_{j+1}}{8}\right) \quad (13)$$

For the linear trial function approximation for h on Ω_j

$$\left(K_s h\frac{\partial h}{\partial x}\right)\bigg|_{x'} = K_s h'\frac{(h_j-h_{j-1})}{(x_j-x_{j-1})} \quad (14)$$

$$\left(K_s h\frac{\partial h}{\partial x}\right)\bigg|_{x''} = K_s h''\frac{(h_{j+1}-h_j)}{(x_{j+1}-x_j)} \quad (15)$$

in which $K_s h'\,(h_j - h_{j-1})$ and $K_s h''\,(h_{j+1} - h_j)$ are constant for all time due to the numerical approximation solving for specified spatial coordinates x_j (Fig. 2). Combining Eqs. 7, 13, 14, and 15 gives the subdomain integration numerical model for Eq. 2 on Ω_j:

$$\frac{8\hat{K}_s h''(h_{j+1}-h_j)}{(x_{j+1}-x_j)} - \frac{8\hat{K}_s h'(h_j-h_{j-1})}{(x_j-x_{j-1})} = \frac{dx_{j-1}}{dt}(h_{j-1}-h_j)$$

$$+ 3\frac{dx_j}{dt}(h_{j-1} - h_{j+1}) + \frac{dx_{j+1}}{dt}(h_j - h_{j+1}) \quad (16)$$

in which $\hat{K}_s = K_s / n$.

For problems where

$$h_{j+1} - h_j = h_j - h_{j-1} = \Delta h \quad (17)$$

and $$\frac{dx_{j-1}}{dt} + \frac{dx_{j+1}}{dt} \simeq 2\frac{dx_j}{dt} \quad (18)$$

the numerical statement of Eq. 16 can be simplified as

$$\frac{\hat{K}_s h'' \Delta h}{(x_{j+1} - x_j)} - \frac{\hat{K}_s h' \Delta h}{(x_j - x_{j-1})} = \Delta h \frac{dx_j}{dt} \quad (19)$$

It may be noted that the K_s parameter has not been assumed constant in the numerical model derivations, i.e., due to the approach of solving for spatial coordinates x_j, nonlinear terms such as $hK(h)$ will be constant on the boundaries of Ω_j.

Equation 17 is integrated with respect to time to give the model approximation for Eq. 2 on Ω_j

$$\int_{t=k\Delta t}^{t=(k+1)\Delta t} \left[\frac{\hat{K}_s h''}{(x_{j+1} - x_j)} - \frac{\hat{K}_s h'}{(x_j - x_{j-1})}\right] dt + x_j(k\Delta t) = x_j [(k+1)\Delta t] \quad (20)$$

Integrating Eq. 20 over a small Δt timestep gives

$$\frac{\hat{K}_s h'' \Delta t}{(x_{j+1}^2 - x_j^2) - (x_{j+1}^1 - x_j^1)} \ln\left(\frac{x_{j+1}^2 - x_j^2}{x_{j+1}^1 - x_j^1}\right)$$
$$- \frac{\hat{K}_s h' \Delta t}{(x_j^2 - x_{j-1}^2) - (x_j^1 - x_{j-1}^1)} \ln\left(\frac{x_j^2 - x_{j-1}^2}{x_j^1 - x_{j-1}^1}\right) + x_j^1 = x_j^2 \quad (21)$$

in which in Eq. 21 the superscripts 1 and 2 designate x-coordinate values at time $k\Delta t$ and $(k + 1)\Delta t$, respectively. Equation 21 can be further simplified by letting

$$\frac{\hat{K}_s h'' \Delta t}{(\bar{x}_{j+1} - \bar{x}_j)} - \frac{\hat{K}_s h' \Delta t}{(\bar{x}_j - \bar{x}_{j-1})} + x_j^1 = x_j^2 \quad (22)$$

in which it is assumed that (12)

$$\bar{x}_j = \frac{1}{2}[3x_j(t = k\Delta t) - x_j(t = k\Delta t - \Delta t) \quad (23)$$

Therefore, an explicit formulation for the approximation of Eq. 2 is developed whereby each future x-coordinate, e.g., x_j^2, can be determined from previously determined data. The development of an equivalent numerical model for Eq. 1 follows analogously to the preceding derivation (6,7,8).

Model Applications

To demonstrate the accuracy of the proposed numerical method, the first problem presented is an idealization of ground-water flow from a confined aquifer that forms the banks of a stream. It is assumed that drawdown is sufficiently small so that Eq. 1 describes the flow process. The initial and boundary conditions assumed in order to simulate an instantaneous step change in the piezometric profile of the aquifer are (Fig. 3)

$$h(x,0) = h_o; \quad h(\infty, t) = h_o, t \geq 0; \quad h(0,t) = 0, t > 0 \qquad (24)$$

The analytical solution to Eqs. 1 and 24 is given by

$$h = h_o \operatorname{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right)$$

in which h_o = the assumed constant step change of stage. For example purposes, the substitutions of a unit step change in stage and $\alpha = 0.25$ (units of L^2/T) were used.

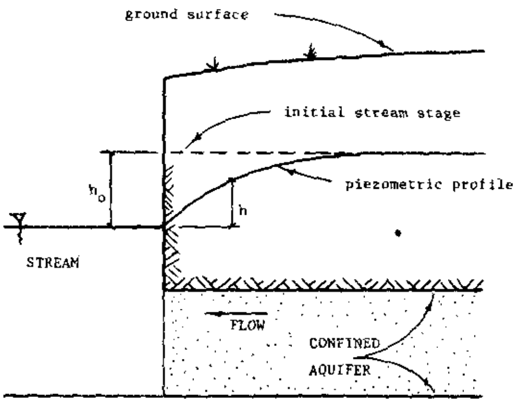

FIG. 3.—Flow from Confined Aquifer to Stream Due to Drop in Stage

Due to the several simplifying assumptions used to reduce the complexity of Eq. 16 to the approximation of Eq. 22, small timestep increments were required. The assumption of Eq. 17 is met by suitable discretization of Ω by equally spaced nodal points. The assumption of Eq. 18, however, limits the time rate of change of the state variable profile, i.e., in the initial portions of the problem solution when the piezometric profile changes relatively rapidly, much smaller timestep increments were required than when the piezometric profile varied more slowly. In order to keep the program size small, several simulations were made using various constant timestep sizes. The resulting modeled profiles were compared until negligible differences in the computed piezometric profiles were observed with decreasing timestep size. The resulting piezometric profile at various intervals of time are compared to model results in Fig. 4. From the figure, good results were obtained in the use of Eq. 22 to model the linear formulation of Eq. 1.

The second problem presented is the application of the model of Eq. 22 to solution of the nonlinear formulation of Eq. 2. The problem considered is

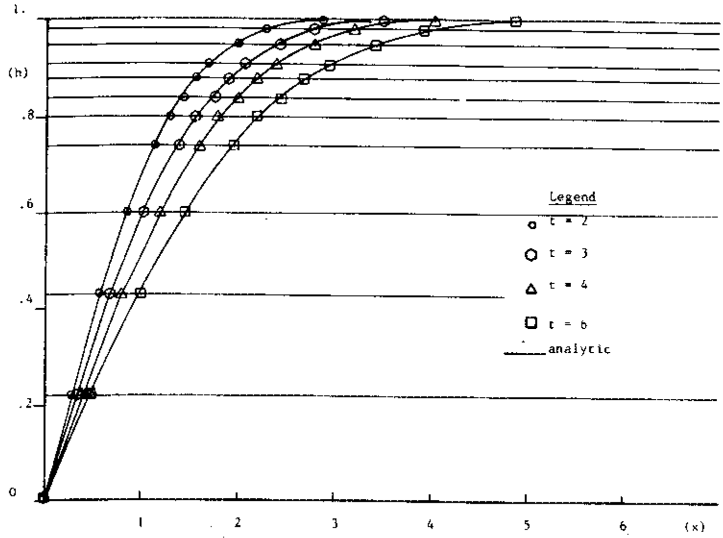

FIG. 4.—Model Results in Predicting Piezometric Profile

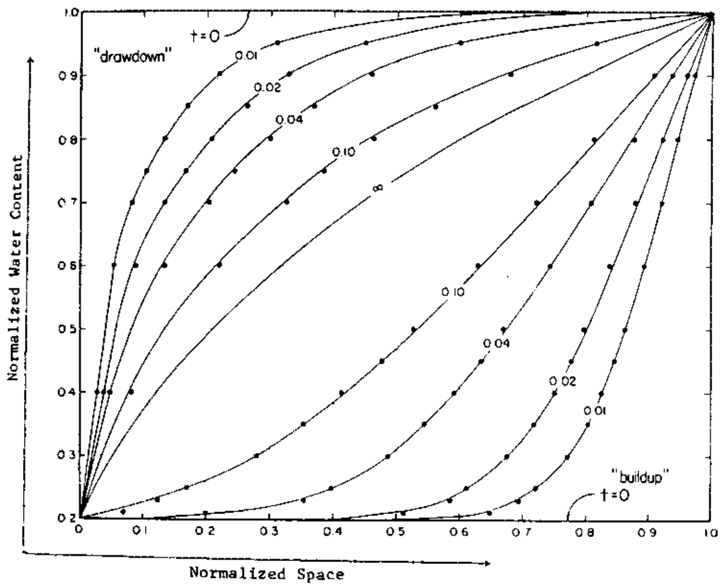

FIG. 5.—Model Results in Predicting Free Water Surface [Dots Represent Modeled Results; Solid Line Represents Analytic Solution to Equation 2 (3)]

the estimation of unconfined ground-water surface profiles for an instantaneous step change in elevation between two reservoirs separated by a given length of soil. Figure 5 shows dimensionless water surface profiles and computed results. For the normalized problems considered, time steps of .000125 were used requiring a total of 720 cycles to advance the profiles from normalized time of 0.01–0.10. As can be seen from Fig. 5, good results are achieved. As with the previous test problems, several simulations were made reducing constant timestep sizes until negligible differences in the computed ground-water surface profiles were observed.

The computational algorithm can be accommodated on current programmable calculators when the simplifying assumptions of Eqs. 18 and 19 are used. A further simplification of the model is the elimination of the time-step approximation of Eq. 23, further reducing calculator memory requirements.

Conclusions

The subdomain integration version of the weighted residuals method is applied to the linear and nonlinear equations of one-dimensional confined and unconfined ground-water flow, respectively. The approach used is to determine the spatial coordinates as a function of piezometric or free ground-water surface profiles. A simplified version of the numerical model can be accommodated by current programmable calculators. Good results were obtained when applying the proposed numerical model to the problems considered. However, several simulations were required for each problem tested, progressively reducing timestep sizes until negligible differences in computed results were observed.

Acknowledgments

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Appendix II.—Notation

The following symbols are used in this paper:

h = piezometric or free water surface elevation;
h_j = nodal point value of h;
K_s = saturated hydraulic conductivity;
$\hat{K}_s$ = K_s/n;
n = specific porosity;
x, t = space and time coordinates;
erf = error function;
α = hydraulic diffusivity;
Δt = time step;
Ω = domain of definition; and
Ω_j = subdomain j.