A note on time integration of unsaturated soil–moisture transport

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The non-linear soil–moisture diffusivity model can be approximately linearized by using values of diffusivity assumed constant for small intervals of space and time. By a series expansion of the diffusivity function and integrating the resulting series of differential equations with respect to time, an improved numerical model is developed. Results from application of this new approach to a sharp wetting-front soil infiltration problem indicates that a 67% saving in numerical effort is achieved at comparable estimation accuracy levels when using the traditional finite timestep Crank–Nicolson approach.

INTRODUCTION

The study of numerical methods for the approximation of non-linear soil–moisture transport in a horizontal column has received recent attention. Hayhoe[2] compared the numerical effectiveness between the finite element and finite difference numerical methods in modelling a sharp wetting-front soil infiltration problem. A special finite difference analog was advanced as the best numerical approach to the problem studied. Hromadka and Guymon[5] further studied the sharp wetting-front problem and developed a modification to the finite element method which resulted in an increase in model accuracy for a linear soil–water diffusivity problem. For a non-linear diffusivity problem, the traditional finite element formulation gave superior results to Hayhoe's finite difference approach when the finite element analog used constant element diffusivity values as determined by a spatial estimation procedure[6].

In this paper, the non-linear soil–water diffusivity problem is re-examined with respect to the finite element modified procedure which we call the 'nodal integration method'. By expanding the diffusivity function as a Taylor series time expansion, the governing equations can be temporally integrated. This procedure results in a numerical analog similar to the nodal integration formulation, but with a constant element diffusivity value determined by a temporal integration. This new numerical approach enables the finite timestep increment to be increased and yet retain similar numerical approximation accuracy. Although the quasi-constant values of diffusivity used to linearize the mathematical model require additional computational effort, the overall computer execution costs are reduced. Reduced costs are achieved because of reduction in the number of finite timestep advancements of the problem's global matrices.

The mathematical development of this numerical procedure is presented herein. As a case study, the integration procedure is used to approximate the numerically difficult problem previously studied by Hayhoe[2] and Hromadka and Guymon[5].

MATHEMATICAL DEVELOPMENT

The one-dimensional horizontal soil–moisture transport model for an unsaturated soil column is:

$$\frac{\partial}{\partial x}\left[D\frac{\partial\theta}{\partial x}\right]=\frac{\partial\theta}{\partial t};\quad x\in R \qquad R\equiv\{x|0\leqslant x\leqslant L\} \tag{1}$$

where θ is the volumetric water content (θ less than the soil's porosity); x is the spatial coordinate; t is time; D is the soil–water diffusivity and is a function of soil–water content; and R is the spatial domain of definition.

The domain R can be discretized by n nodal points $\theta_j (j=1, 2, \ldots, n)$ into n disjoint subsets:

$$\begin{aligned}R_1&\equiv\{x|0\leqslant x\leqslant(x(\theta_1)+x(\theta_2))/2\}\\ R_2&\equiv\{x|(x(\theta_1)+x(\theta_2))/2<x\leqslant(x(\theta_2)+x(\theta_3))/2\}\\ &\vdots\\ R_n&\equiv\{x|(x(\theta_{n-1})+x(\theta_n))/2<x\leqslant x(\theta_n)=L\}\end{aligned} \tag{2}$$

where $x(\theta_j)$ is the spatial coordinate associated to nodal point θ_j, and

$$R=\bigcup_{j=1}^{n}R_j \tag{3}$$

Equation (1) must be satisfied on each R_j. Therefore, n equations are generated by solving:

$$\frac{\partial}{\partial x}\left[D\frac{\partial\theta}{\partial x}\right]=\frac{\partial\theta}{\partial t};\quad x\in R_j,\ \forall_j \tag{4}$$

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where

$$D = D(\theta)$$
$$\theta = \theta(x,t) \tag{5}$$

Integrating equation (4) with respect to space gives:

$$\left\{D\frac{\partial\theta}{\partial x}\right\}\bigg|_{\Gamma_j} = \frac{\partial}{\partial t}\int_{R_j}\theta \mathrm{d}x; \qquad x \in R_j, \forall_j \tag{6}$$

where Γ_j is the spatial boundary of region R_j. Integrating equation (6) with respect to time gives:

$$\int_{k\Delta t}^{(k+1)\Delta t}\left\{D\frac{\partial\theta}{\partial x}\right\}\bigg|_{\Gamma_j}\mathrm{d}t = \int_{R_j}\{\theta\}\bigg|_{\Gamma_t}\mathrm{d}x \tag{7}$$

where Γ_t is the limits of temporal integration between timesteps $k\Delta t$ and $(k+1)\Delta t$. Equation (7) can be simplified by using the linear transformation:

$$t = k\Delta t + \varepsilon$$
$$0 \leqslant \varepsilon \leqslant \Delta t \tag{8}$$

Thus,

$$\int_0^{\Delta t}\left\{D(k\Delta t+\varepsilon)\frac{\partial\theta(k\Delta t+\varepsilon)}{\partial x}\right\}\bigg|_{\Gamma_j}\mathrm{d}\varepsilon = \int_{R_j}\{\theta\}\bigg|_{\Gamma_t}\mathrm{d}x \tag{9}$$

The soil–water diffusivity function can be expressed with respect to time by the Taylor series:

$$D(x=x_0, k\Delta t+\varepsilon) = \sum_{i=0}^{\infty}\frac{D^{(i)}(x=x_0, k\Delta t)\varepsilon^i}{i!} \tag{10}$$

where i is the ith order temporal differential operator; and x_0 is a specified spatial coordinate. Combining equations (9) and (10) gives:

$$\int_0^{\Delta t}\left\{\left(\sum_{i=0}^{\infty}\frac{D^{(i)}(k\Delta t)\varepsilon^i}{i!}\right)\frac{\partial\theta(k\Delta t+\varepsilon)}{\partial x}\right\}\bigg|_{\Gamma_j}\mathrm{d}\varepsilon = \int_{R_j}\{\theta\}\bigg|_{\Gamma_t}\mathrm{d}x \tag{11}$$

For a spatial local coordinate system defined by:

$$y = x - (x(\theta_j)+x(\theta_{j-1}))/2 \quad x \in R_j$$
$$\mathrm{d}y = \mathrm{d}x$$
$$l_j = (x(\theta_{j+1}) - x(\theta_{j-1}))/2 \tag{12}$$

Equation (11) can be expanded as:

$$\sum_{i=0}^{\infty}\frac{D^{(i)}(y=l_j, k\Delta t)}{i!}\int_0^{\Delta t}\varepsilon^i\left\{\frac{\partial\theta(k\Delta t+\varepsilon)}{\partial y}\right\}\bigg|_{y=l_j}\mathrm{d}\varepsilon -$$
$$\sum_{i=0}^{\infty}\frac{D^{(i)}(y=0, k\Delta t)}{i!}\int_0^{\Delta t}\varepsilon^i\left\{\frac{\partial\theta(k\Delta t+\varepsilon)}{\partial y}\right\}\bigg|_{y=0}\mathrm{d}\varepsilon$$
$$= \int_{R_j}\{\theta\}\bigg|_{\Gamma_t}\mathrm{d}y \tag{13}$$

The soil–water content function is approximated spatially and temporally by:

$$\theta \simeq \hat{\theta}$$
$$\hat{\theta} = \sum_{r=1}^{n}N_r\left(\sum_{m=0}^{k+1}M_m\,\theta_r^m\right) \tag{14}$$

where N_r and M_m are the linearly independent spatial and temporal shape functions, and

$$\theta_r^m = \hat{\theta}(x=x(\theta_r), t=m\,\Delta t) \tag{15}$$

where the θ_r^m are known values for time steps $m = 0, 1, \ldots, k$. The spatial gradient of the soil–water content function is approximated by:

$$\frac{\partial\theta}{\partial x} \simeq \frac{\partial\hat{\theta}}{\partial x} = \sum_{r=1}^{n}\frac{\partial N_r}{\partial x}\left(\sum_{m=0}^{k+1}M_m\,\theta_r^m\right) \tag{16}$$

Substituting equations (14) and (16) into equation (13) gives the numerical approximation:

$$\sum_{i=0}^{\infty}\frac{D^{(i)}(y=l_j, k\Delta t)}{i!}\int_0^{\Delta t}\varepsilon^i\left\{\sum_{r=1}^{n}\frac{\partial N_r}{\partial x}\left(\sum_{m=0}^{k+1}M_m\,\theta_r^m\right)\right\}\bigg|_{y=l_j}\mathrm{d}\varepsilon -$$
$$\sum_{i=0}^{\infty}\frac{D^{(i)}(y=0, k\Delta t)}{i!}\int_0^{\Delta t}\varepsilon^i\left\{\sum_{r=1}^{n}\frac{\partial N_r}{\partial x}\left(\sum_{m=0}^{k+1}M_m\,\theta_r^m\right)\right\}\bigg|_{y=0}\mathrm{d}\varepsilon$$
$$= \int_0^{l_j}\left\{\sum_{r=1}^{n}N_r\left(\sum_{m=0}^{k+1}M_m\,\theta_r^m\right)\right\}\bigg|_{\Gamma_t}\mathrm{d}y \tag{17}$$

The unknown values of nodal points θ_r^{k+1} can be solved by equating:

$$\sum_{i=0}^{\infty}\frac{D^{(i)}(y=l_j, k\Delta t)}{i!}\int_0^{\Delta t}\varepsilon^i\left\{\sum_{r=1}^{n}\frac{\partial N_r}{\partial x}M_{k+1}\,\theta_r^{k+1}\right\}\bigg|_{y=l_j}\mathrm{d}\varepsilon -$$
$$\sum_{i=0}^{\infty}\frac{D^{(i)}(y=0, k\Delta t)}{i!}\int_0^{\Delta t}\varepsilon^i\left\{\sum_{r=1}^{n}\frac{\partial N_r}{\partial x}M_{k+1}\,\theta_r^{k+1}\right\}\bigg|_{y=0}\mathrm{d}\varepsilon -$$
$$\int_0^{l_j}\sum_{r=1}^{n}N_r\,\theta_r^{k+1}\,\mathrm{d}x =$$
$$\sum_{i=0}^{\infty}\frac{D^{(i)}(y=0, k\Delta t)}{i!}\int_0^{\Delta t}\varepsilon^i\left\{\sum_{r=1}^{n}\frac{\partial N_r}{\partial x}\left(\sum_{m=0}^{k}M_m\,\theta_r^m\right)\right\}\bigg|_{y=0}\mathrm{d}\varepsilon -$$

$$\sum_{i=0}^{\infty}\frac{D^{(i)}(y=l_j,k\Delta t)}{i!}\int_0^{\Delta t}\varepsilon^i\left\{\sum_{r=1}^{n}\frac{\partial N_r}{\partial x}\left(\sum_{m=0}^{k}M_m\,\theta_r^m\right)\right\}\bigg|_{y=l_j}\mathrm{d}\varepsilon-\int_0^{l_j}\sum_{r=1}^{n}N_r\,\theta_r^k\,\mathrm{d}x \qquad (18)$$

NUMERICAL MODEL

The space–time surface approximated by equation (14) can be simplified by assuming that the functional surface can be described by sets of piecewise continuous polynomials. For a parabolic spatial shape function approximation for θ between nodal points $(\theta_{j-1},\theta_j,\theta_{j+1})$.

$$\frac{\partial\theta}{\partial x}\bigg|_{y=l}=(\theta_{j+1}-\theta_j)/l$$

$$\frac{\partial\theta}{\partial x}\bigg|_{y=0}=(\theta_j-\theta_{j-1})/l \qquad (19)$$

$$\int_{y=0}^{l}\theta\,\mathrm{d}y=\frac{l}{24}[\theta_{j-1}+22\theta_j+\theta_{j+1}]$$

where for discussion purposes it is assumed that in equation (19)

$$x(\theta_{j+1})-x(\theta_j)=x(\theta_j)-x(\theta_{j-1})=l$$

$$\mathrm{d}x=\mathrm{d}y$$

For a linear polynomial function approximation for the time curves between time-steps $(k,k+1)$ where $(k+1)$ is the time step to be evaluated,

$$\theta_j(k\Delta t+\varepsilon)=(\theta_j^{k\Delta t})\left(\frac{\Delta t-\varepsilon}{\Delta t}\right)+(\theta_j^{(k+1)\Delta t})\frac{\varepsilon}{\Delta t}$$

$$k\Delta t\leqslant t\leqslant(k+1)\,\Delta t \qquad (20)$$

$$\mathrm{d}\varepsilon=\mathrm{d}t$$

Combining equations (19) and (20), the spatial gradient approximation during the time-step Δt is given by:

$$\frac{\partial\hat\theta}{\partial x}\bigg|_{y=l}=(\theta_{j+1}^2-\theta_{j+1}^1-\theta_j^2+\theta_j^1)\varepsilon/l\Delta t+(\theta_{j+1}^1-\theta_j^1)/l$$

$$\frac{\partial\hat\theta}{\partial x}\bigg|_{y=l}=(\theta_j^2-\theta_j^1-\theta_{j-1}^2+\theta_{j-1}^1)\varepsilon/l\Delta t+(\theta_j^1-\theta_{j-1}^1)/l$$

$$0\leqslant\varepsilon\leqslant\Delta t \qquad (21)$$

where superscripts 1 and 2 refer to timesteps $k\Delta t$ and $(k+1)\Delta t$, respectively. Combining equations (18), (19) and (21) gives:

$$\sum_{i=0}^{\infty}\frac{D^{(i)}(y=l)}{i!}\int_0^{\Delta t}\left\{(\theta_{j+1}^2-\theta_{j+1}^1-\theta_j^2+\theta_j^1)\frac{\varepsilon^{i+1}}{l\Delta t}+(\theta_{j+1}^1-\theta_j^1)\frac{\varepsilon^i}{l}\right\}\mathrm{d}\varepsilon-$$

$$\sum_{i=0}^{\infty}\frac{D^{(i)}(y=0)}{i!}\int_0^{\Delta t}\left\{(\theta_j^2-\theta_j^1-\theta_{j-1}^2+\theta_{j-1}^1)\frac{\varepsilon^{i+1}}{l\Delta t}+(\theta_j^1-\theta_{j-1}^1)\frac{\varepsilon^i}{l}\right\}\mathrm{d}\varepsilon$$

$$=\frac{l}{24}[(\theta_{j-1}^2+22\theta_j^2+\theta_{j+1}^2)-(\theta_{j-1}^1+22\theta_j^1+\theta_{j+1}^1)] \qquad (22)$$

The temporal integration of equation (22) is evaluated by isolating the time-integrated function as:

$$\sum_{i=0}^{\infty}\frac{D^{(i)}(y=l)}{i!}\left[\frac{(\theta_{j+1}^2-\theta_j^2)-(\theta_{j+1}^1-\theta_j^1)}{l\Delta t}\right]\int_0^{\Delta t}\varepsilon^{i+1}\,\mathrm{d}\varepsilon+$$

$$\sum_{i=0}^{\infty}\frac{D^{(i)}(y=l)}{i!}\left[\frac{\theta_{j+1}^1-\theta_j^1}{l}\right]\int_0^{\Delta t}\varepsilon^i\mathrm{d}\varepsilon-$$

$$\sum_{i=0}^{\infty}\frac{D^{(i)}(y=0)}{i!}\left[\frac{(\theta_j^2-\theta_{j-1}^2)-(\theta_j^1-\theta_{j-1}^1)}{l\Delta t}\right]\int_0^{\Delta t}\varepsilon^{i+1}\,\mathrm{d}\varepsilon-$$

$$\sum_{i=0}^{\infty}\frac{D^{(i)}(y=0)}{i!}\left[\frac{\theta_j^1-\theta_{j-1}^1}{l}\right]\int_0^{\Delta t}\varepsilon^i\mathrm{d}\varepsilon=$$

$$\frac{l}{24}[(\theta_{j-1}^2+22\theta_j^2+\theta_{j+1}^2)-(\theta_{j-1}^1+22\theta_j^1+\theta_{j+1}^1)] \qquad (23)$$

Rearranging terms, the nodal point expressions can be isolated by:

$$[\theta_{j+1}^2-\theta_j^2]\sum_{i=0}^{\infty}\frac{D^{(i)}(y=l)}{i!l\Delta t}\int_0^{\Delta t}\varepsilon^{i+1}\,\mathrm{d}\varepsilon-$$

$$[\theta_j^2-\theta_{j-1}^2]\sum_{i=0}^{\infty}\frac{D^{(i)}(y=0)}{i!l\Delta t}\int_0^{\Delta t}\varepsilon^{i+1}\,\mathrm{d}\varepsilon+$$

$$[\theta_{j+1}^1-\theta_j^1]\sum_{i=0}^{\infty}\frac{D^{(i)}(y=l)}{i!l\Delta t}\int_0^{\Delta t}(\Delta t\varepsilon^i-\varepsilon^{i+1})\,\mathrm{d}\varepsilon-$$

$$[\theta_j^1-\theta_{j-1}^1]\sum_{i=0}^{\infty}\frac{D^{(i)}(y=0)}{i!l\Delta t}\int_0^{\Delta t}(\Delta t\varepsilon^i-\varepsilon^{i+1})\,\mathrm{d}\varepsilon=$$

$$\frac{l}{24}[\theta_{j-1}^2+22\theta_j^2+\theta_{j+1}^2]-\frac{l}{24}[\theta_{j-1}^1+22\theta_j^1+\theta_{j+1}^1] \qquad (24)$$

Carrying out the indicated integration in equation (24) gives:

$$[\theta_{j+1}^2-\theta_j^2]\sum_{i=0}^{\infty}\frac{D^{(i)}(y=l)(\Delta t)^{i+1}}{i!l\quad(i+2)}-$$

$$[\theta_j^2-\theta_{j-1}^2]\sum_{i=0}^{\infty}\frac{D^{(i)}(y=0)(\Delta t)^{i+1}}{i!l\quad(i+2)}+$$

$$[\theta_{j+1}^1-\theta_j^1]\sum_{i=0}^{\infty}\frac{D^{(i)}(y=l)\quad(\Delta t)^{i+1}}{i!l\quad(i+1)(i+2)}-$$

$$[\theta_j^1-\theta_{j-1}^1]\sum_{i=0}^{\infty}\frac{D^{(i)}(y=0)\quad(\Delta t)^{i+1}}{i!l\quad(i+1)(i+2)}=$$

$$\frac{l}{24}[\theta_{j-1}^2+22\theta_j^2+\theta_{j+1}^2]-\frac{l}{24}[\theta_{j-1}^1+22\theta_j^1+\theta_{j+1}^1] \tag{25}$$

In a different notation, equation (25) can be rewritten as:

$$\bar{D}_l[\theta_{j+1}^2-\theta_j^2]-\bar{D}_0[\theta_j^2-\theta_{j-1}^2]-\frac{l}{24}[\theta_{j-1}^2+22\theta_j^2+\theta_{j+1}^2]$$

$$=-\bar{\bar{D}}_l[\theta_{j+1}^1-\theta_j^1]+\bar{\bar{D}}_0[\theta_j^1-\theta_{j-1}^1]-\frac{l}{24}[\theta_{j-1}^1+22\theta_j^1+\theta_{j+1}^1] \tag{26}$$

where

$$\bar{D}_\eta=\sum_{i=0}^{\infty}\frac{D^{(i)}(y=\eta)(\Delta t)^{i+1}}{i!(i+2)\quad l};\qquad \eta=0,\ l$$

$$\bar{\bar{D}}_\eta=\sum_{i=0}^{\infty}\frac{D^{(i)}(y=\eta)(\Delta t)^{i+1}}{(i+2)!l};\qquad \eta=0,\ l \tag{27}$$

Since the space–time surface is assumed to be linear with respect to time, the temporal differentials of soil–water diffusivity in equation (27) are given by:

$$D^{(N)}\equiv\frac{\partial^N D}{\partial t^N}=\frac{\partial^N D}{\partial\theta^N}\left(\frac{\partial\theta}{\partial t}\right)^N \tag{28}$$

where the temporal water content gradient can be approximated from values of water content during time step k.

Hromadka and Guymon[6] determine a complete formulation for a parabolic spatial interpolation function which involved five nodal points rather than three in order to estimate nodal point values; however, the additional computational effort did not significantly increase the approximation accuracy. In their study, the (linear interpolation shape function) Galerkin analog to equation (1), was determined and modified to correspond to equation (26) for the special case of $i=0$.

The Galerkin version of the weighted residual process can be used to approximate equation (1) by the finite element method. The solution domain is discretized into the union of n finite elements by:

$$L=\bigcup_{i=1}^{n}L_i \tag{29}$$

The water content is utilized as the state variable and is approximated within each finite element by:

$$\theta(x)=\Sigma N_j(x)\theta_j \tag{30}$$

where N_j = the appropriate linearly independent shape functions; θ_j = state variable values at element-nodal points designated by the general summation index j.

The Galerkin technique utilizes the set of shape functions as the weighting functions, which indicates that the corresponding finite element representation for the infiltration process is:

$$\iint\left\{\frac{\partial}{\partial x}\left[D(\theta)\frac{\partial\theta}{\partial x}\right]-\frac{\partial\theta}{\partial t}\right\}N_j\mathrm{d}x=0 \tag{31}$$

Integration by parts expands equation (31) into the form:

$$\sum_{i=1}^{n}\left\{D(\theta)\frac{\partial\theta}{\partial x}N_j\Big|_{S_i}-\int_{L_i}\left[D(\theta)\frac{\partial\theta\partial N_j}{\partial x\ \partial x}+N_j\frac{\partial\theta}{\partial t}\right]\mathrm{d}x\right\}=0 \tag{32}$$

where S_i = external endpoints of the one-dimensional finite element, L_i. The first term within the braces sums to zero for interior elements, and also satisfies the usual specified (or flux type) boundary conditions of the problem for exterior finite elements. The remaining integral term is solved by substituting the appropriate element approximations and shape functions into the integrand and solving by numerical integration. A convenient approach to deal with the non-linearity of equation (32) is to assume the diffusivity function to be constant within each finite element during a finite time interval, Δt, in order to carry out the integration. The Crank–Nicolson time advancement approximation has been widely used[1,2] to solve the time derivative of equation (32).

The Crank–Nicolson formulation reduces equation (32), where values of soil–water diffusivity are assumed constant within each finite element, into a system of linear equations expressed in matrix form as:

$$\left\{\mathbf{P}+\frac{\Delta t}{2}\mathbf{S}\right\}\boldsymbol{\theta}^{i+1}=\left\{\mathbf{P}-\frac{\Delta t}{2}\mathbf{S}\right\}\boldsymbol{\theta}^{i} \tag{33}$$

where $\mathbf{P}$ is a symmetrical capacitance matrix and is a function of element nodal global coordinates; $\mathbf{S}$ is a symmetrical stiffness matrix and is a function of element nodal global coordinates and constant finite element diffusivity coefficients (during time step Δt); Δt is the finite time step increment; and $\boldsymbol{\theta}^k$ is the vector of nodal state variable approximations (volumetric water content) at time steps $k=i,\ i+1$.

For a linear polynomial shape function, the element matrices determined from equation (32) are given by:

$$\hat{\mathbf{S}}\begin{Bmatrix}\theta_i\\ \theta_j\end{Bmatrix}+\hat{\mathbf{P}}\begin{Bmatrix}\dot\theta_i\\ \dot\theta_j\end{Bmatrix}=\frac{D_i}{l_i}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\begin{Bmatrix}\theta_i\\ \theta_j\end{Bmatrix}+\frac{l_i}{6}\begin{bmatrix}2&1\\1&2\end{bmatrix}\begin{Bmatrix}\dot\theta_i\\ \dot\theta_j\end{Bmatrix} \tag{34}$$

where D_i is the quasi-constant diffusivity within element i; $\hat{\mathbf{S}}$ and $\hat{\mathbf{P}}$ are element stiffness and capacitance matrices, respectively; and $(\theta_i,\ \theta_j)$ and $(\dot\theta_i,\ \dot\theta_j)$ refer to the element nodal and dynamic nodal moisture content values, respectively, for an element of length l_i.

For the linear temporal interpolation function, equations (26) and (27) can be written analogously to equations (33) and (34) as:.

$$\{\bar{\mathbf{P}}+\bar{\mathbf{S}}\}\boldsymbol{\theta}^{i+1}=\{\bar{\mathbf{P}}-\bar{\bar{\mathbf{S}}}\}\boldsymbol{\theta}^{i} \tag{35}$$

Table 1. Comparison of numerical efficiency at time t = 16.5 minutes (values of water content)

x (cm)	Exact (Hayhoe)	$\Delta t = 0.1$ min $i=0$	$\Delta t = 0.1$ min $i=2$	$\Delta t = 0.1$ min $i=5$	$\Delta t = 0.3$ min $i=0$	$\Delta t = 0.3$ min $i=2$	$\Delta t = 0.3$ min $i=5$
0.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
0.5	0.99	0.99	0.99	0.99	0.98	0.99	0.99
1.0	0.97	0.97	0.97	0.97	0.97	0.97	0.97
1.5	0.95	0.95	0.95	0.95	0.94	0.95	0.95
2.0	0.92	0.93	0.92	0.92	0.92	0.93	0.93
2.5	0.88	0.90	0.89	0.89	0.88	0.90	0.90
3.0	0.84	0.87	0.85	0.85	0.84	0.87	0.86
3.5	0.78	0.82	0.78	0.78	0.82	0.82	0.82
4.0	0.67	0.39	0.63	0.64	0.05	0.37	0.40
4.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

i: number of temporal Taylor series terms (diffusivity function) included
Δt: time step magnitude

where the element matrices composing the global **S** matrices of equation (35) are given by:

$$\bar{\bar{\mathbf{S}}} \equiv \bar{\bar{D}}_n \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\bar{\mathbf{S}} \equiv \bar{D}_n \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{36}$$

and

$$\bar{\mathbf{P}} \equiv \frac{l}{24} \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix} \tag{37}$$

MODEL APPLICATION

The numerical model given by equations (35), (36) and (37) was applied to a sharp wetting-front problem of soil–water infiltration into an air-dry horizontal column[2,5]. The analytic value of soil–water diffusivity for Hanford sandy loam[4] was selected in order to provide a sharp wetting front through the soil column, causing the numerical analysis of moisture flow in the soil to be difficult. The quasi-analytic solution advanced by Philip and Knight[3] and utilized by Hayhoe[2] was used for this study.

Equation (1) was solved subject to the initial condition:

$$\theta(x,t) = 0;\ t = 0,\ 0 \leqslant x \leqslant L \tag{38}$$

and the boundary conditions:

$$\theta(0,\ t) = 1,\ \theta(L,\ t) = 0;\ t > 0 \tag{39}$$

where the soil–water diffusivity (cm^2 min^{-1}) is given by:

$$D(\theta) = \begin{cases} 0.9 \times 10^{-3} \exp(8.36\theta), & \theta > 0 \\ 0.9 \times 10^{-3}, & \theta = 0 \end{cases} \tag{40}$$

and θ is the dimensionless volumetric water content.

Owing to the nature of the soil–water diffusivity function of (40), the temporal variation of diffusivity is extremely important during the Δt time step. As such, the temporal diffusivity Taylor series finds good use in sharp wetting front infiltration problems as presented herein. For moisture transport problems where the diffusivity function causes less difficulty in numerical estimation, the model represented by equation (35) can be used to reduce execution time by approximately two-thirds. That is, by expanding the time series of equation (27), the time step magnitude Δt was found to be capable of a 300% increase and yet retain the same level of approximation accuracy. Table 1 contains various values of time step magnitudes Δt (in min) and time series expansion terms in the numerical model of equation (1) by (35) at time $t = 16.5$ min. For the numerical model, a spatial discretization of 0.5 cm was used where the total column length was set at 5.0 cm to correspond to the model results of Hayhoe[2].

The test results indicate that the numerical accuracy achieved for $\Delta t = 0.1$ min (without the temporal Taylor series expansion for diffusivity) is also matched for a $\Delta t = 0.3$ min time step where the temporal Taylor series is included through the fifth term. Since the integration of the diffusivity series expansion involves negligible computer execution time when compared to the solution of matrices during the time advancement, further increases in time step sizes may be possible for less difficult models.

CONCLUSIONS

A new numerical approach to soil water (diffusivity) infiltration problems is advanced. By use of a modified finite element analog to the governing equations, a Taylor series of soil water diffusivity with respect to time can be integrated. Application to a case study indicates a reduction in execution costs due to the capability of increasing the finite time step and yet retain comparable numerical accuracy.

The inclusion of the Taylor series expansion for diffusivity (10) in the traditional finite element matrix formulation (34) was found to produce less desirable results. Using the numerical model for equation (1) given in Hromadka and Guymon[5], the addition of the temporal Taylor series modification produced an increasing over-estimation of the wetting front advancement into the horizontal soil column. That is, as the number of the diffusivity function Taylor series terms were increased in equation (34), the numerical solution indicated an increasingly inaccurate penetration of moisture into the horizontal soil column. This numerical behaviour is probably due to the finite element capacitance matrix over-estimation of mean moisture content within the nodal domains[5].

A sophistication of the proposed integration technique is to use higher order polynomial interpolation functions for the time variable. For a parabolic interpolating function, the increase in numerical accuracy was found to be negligible[6]. However, some advantage was found when using a parabolic interpolation for time in order to estimate the temporal gradient of the diffusivity function (28). For the horizontal soil column, the timestep increase (at comparable numerical accuracy) was found to be approximately 330%, whereas the linear estimation procedure enabled a 300% increase in timestep magnitude. The additional computer memory and computation requirements involving a parabolic temporal interpolation, however, somewhat offsets the advantage of the additional timestep increase in magnitude.

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