Research Communication: Computer interaction and the CVBEM in engineering design

T. V. HROMADKA II

Research Engineer, University of California, Irvine, California, USA

The Complex Variable Boundary Element Method or CVBEM can be used in a computer-aided-design environment wherein the numerical analyst specifies additional boundary element nodal point locations based on computed errors in satisfying the problem boundary conditions. In this fashion, the analyst develops a problem geometry which is acceptable for the construction design, and the CVBEM determines the exact solution for potential problems redefined over this new geometry. Because the computer interactive technique uses graphical displays, the approach is efficient, and easy to use. Additionally, accurate solutions to potential problems can be obtained without the need for special training in the theory or use of the CVBEM.

INTRODUCTION

The Complex Variable Boundary Element Method or CVBEM has been shown to be a useful tool for the numerical analysis of Laplace or Poisson equation boundary value problems.[1] The numerical procedure is to discretize the boundary Γ by nodal points into boundary elements, and then specify a continuous global trial function $G(\zeta)$ on Γ as a function of the nodal values. Using the Cauchy integral, the resulting integral equation is

$$\hat{\omega}(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{G(\zeta)\, d\zeta}{\zeta - z_0} \tag{1}$$

where $\hat{\omega}(z_0)$ is the CVBEM approximation for $z_0 \in \Omega$; and Ω is a two-dimensional simply connected domain enclosed by the simple closed contour Γ.

Because $G(\zeta)$ is continuous on Γ, then $\hat{\omega}(z)$ is analytic over Ω and can be rewritten as the sum of two harmonic functions

$$\hat{\omega}(z) = \hat{\phi}(z) + i\hat{\psi}(z) \tag{2}$$

Thus both $\hat{\phi}(z)$ and $\hat{\psi}(z)$ exactly satisfy the Laplace equation over Ω.

Approximation error occurs due to $\hat{\omega}(z)$ not satisfying the boundary conditions on Γ exactly. However, an approximative boundary $\hat{\Gamma}$ can be developed (by trial and error) which represents the location of points where $\hat{\omega}(z)$ does equal the specified boundary conditions such as level curves (see Fig. 1). Consequently, the CVBEM approximation error can be interpreted as a transformation of $\Gamma \to \hat{\Gamma}$ where the ultimate objective is to have $\hat{\Gamma}$ coincident with Γ. Because all the error of approximation is due to the incorrect boundary element trial functions, accuracy is increased by the addition of boundary nodal points where approximation error is large (i.e. adaptive integration).

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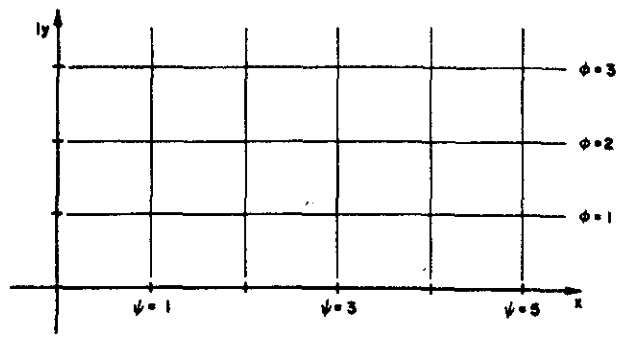

Figure 1. Level curves of an analytic function (example shown: $w(z) = z$)

In this paper, a computer interactive technique is reported which graphically displays Γ and $\hat{\Gamma}$ so that the numerical analyst can readily specify additional nodal points on the CRT screen. In this fashion, the user interacts with the CVBEM to locate the necessary nodal point additions until $\hat{\Gamma}$ and Γ are within an acceptable level of tolerance. For example, the tolerance may be the allowable construction limits specified for a shaft (torsion problem) for use in aircraft design.

As $\hat{\Gamma}$ approaches Γ geometrically, the analyst is assured by the Maximum Modulus Theorem that the maximum approximation error occurs on Γ and that the governing partial differential equation (Laplace) is solved exactly. Consequently, the final product is the exact solution for a problem geometry which is within the construction tolerance of the design.

THEORETICAL BACKGROUND OF THE CVBEM

A complete presentation of the CVBEM development, case studies, mathematical proofs of convergence and existence, and several FORTRAN computer programs are given in Hromadka.[1] In order to develop the geometric interpreta-

tion of modeling error associated with the approximative boundary concept, a brief development of the CVBEM numerical technique is presented in the following.

Let Ω be a simply connected two-dimensional domain (i.e. no holes within Ω) enclosed by a simple closed contour Γ.[2] Let $\phi(x,y)$ be a two-dimensional harmonic function over $\Omega \cup \Gamma$; that is,

$$\frac{\partial^2\phi(x,y)}{\partial x^2} + \frac{\partial^2\phi(x,y)}{\partial y^2} = 0, \quad (x,y) \in \Omega \cup \Gamma \tag{3}$$

Then there exists a simply connected domain Ω^* such that $\Omega \cup \Gamma$ is a proper subset of Ω^* and $\phi(x,y)$ is harmonic over Ω^*.

There exists a harmonic function $\psi(x,y)$ conjugate to $\phi(x,y)$ which also satisfies the Laplace equation of (3) over Ω^* and additionally satisfies the Cauchy-Riemann conditions of

$$\frac{\partial\phi(x,y)}{\partial x} = \frac{\partial\psi(x,y)}{\partial y}, \quad \frac{\partial\phi(x,y)}{\partial y} = -\frac{\partial\psi(x,y)}{\partial x} \tag{4}$$

Let $z = x + iy$ be a complex variable over Ω^*. Then both $\phi(x,y)$ and $\psi(x,y)$ can be written in terms of $\phi(z)$ and $\psi(z)$ such that an analytic function $\omega(z)$ is defined over Ω^* by

$$\omega(z) = \phi(z) + i\psi(z) \tag{5}$$

where to simply notation, (5) can be rewritten as $\omega = \phi + i\psi$, $z \in \Omega^*$.

Equation (5) represents a relationship between two conjugate harmonic functions generally called the potential (ϕ) and stream functions (ψ). A list of typical potential and stream functions which occur in engineering and physics is given in Table 1.[2]

The Cauchy integral theorem equates values of $\omega(z_0)$ for $z_0 \in \Omega$ to a line integral of $\omega(\zeta)$ for $\zeta \in \Gamma$ by

$$\omega(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(\zeta)\,d\zeta}{\zeta - z} \tag{6}$$

Table 1. Potential and stream functions

Physical phenomenon	$\phi(x,y)$ = constant	$\psi(x,y)$ = constant
Heat flow	Isothermals	Heat flow lines
Electrostatics	*Equipotentials*	*Flux lines*
Fluid flow	Equipotentials	Stream lines
Gravitational flow	Potentials	Lines of force
Magnetism	Potentials	Lines of force
Diffusion	Concentration	Lines of force
Elasticity	Strain	Stress lines
Current flow	Potential	Lines of force

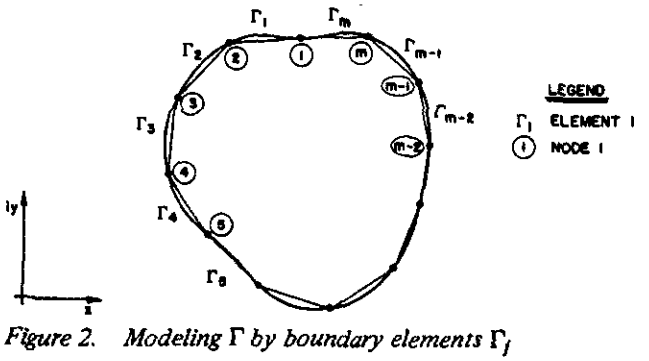

Figure 2. Modeling Γ by boundary elements Γ_j

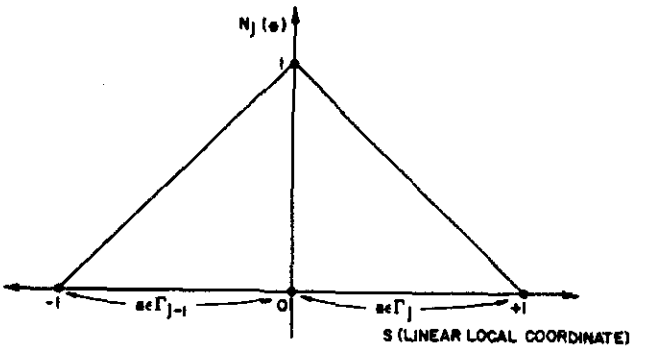

Figure 3. Linear basis function

To illustrate the development of a CVBEM approximation function, $\hat{\omega}(z)$, consider $\omega(z)$ to be defined over Ω^* with $\Omega \cup \Gamma$ interior of Ω^*. Subdivide Γ into m boundary elements Γ_j such as shown in Fig. 2. Nodal points are specified at each element endpoint (here, a linear polynomial CVBEM approximation is being developed). At each node, determine nodal values of $\omega(z)$ by

$$\omega(z_j) \equiv \omega_j = \phi(z_j) + i\psi(z_j) \equiv \phi_j + i\psi_j; \; j = 1, 2, \ldots, m \tag{7}$$

Then a global trial function of $\omega(z)$ is determined for $z \in \Gamma$ by

$$G(z) = \sum_{j=1}^{m} \delta_j [\omega_j N_j(z) + \omega_{j+1} N_{j+1}(z)] \tag{8}$$

where the $N_j(z)$ are linear basis functions (see Fig. 3); and $\delta_j = 1$ for $z \in \Gamma_j$, and $\delta_j = 0$ for $z \notin \Gamma_j$. Substituting $G(z)$ in place of $\omega(\zeta)$ in (6) determines a CVBEM approximation $\hat{\omega}(z)$ of $\omega(z)$

$$\hat{\omega}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{G(\zeta)\,d\zeta}{\zeta - z} \tag{9}$$

Letting $\| \Gamma_m \| = \max | z_{j+1} - z_j |, j = 1, 2, \ldots, m$, then it is seen (without proof) that

$$\lim_{\| \Gamma_m \| \to 0} G(\zeta) = \omega(\zeta), \quad \zeta \in \Gamma \tag{10}$$

and therefore

$$\lim_{\| \Gamma_m \| \to 0} (\omega(z) - \hat{\omega}(z)) = \lim_{\| \Gamma_m \| \to 0} \frac{1}{2\pi i}\int_\Gamma \frac{(\omega(\zeta) - G(\zeta))\,d\zeta}{\zeta - z} = 0 \tag{11}$$

Thus the error of approximation, $e(z)$, is defined by

$$e(z) = \frac{1}{2\pi i}\int_\Gamma \frac{(\omega(\zeta) - G(\zeta))\,d\zeta}{\zeta - z} \tag{12}$$

Because $G(\zeta)$ is continuous on Γ then $\hat{\omega}(z)$ is analytic over Ω which implies both $\hat{\phi}(z)$ and $\hat{\psi}(z)$, where $\hat{\omega}(z) = \hat{\phi}(z) + i\hat{\psi}(z)$, are potential functions over Ω.

In practice, $\phi(z)$ is known on Γ_ϕ and $\psi(z)$ is known on a separate contour on Γ_ψ where $\Gamma = \Gamma_\phi \cup \Gamma_\psi$. Thus $\hat{\omega}(z)$ is not completely defined without estimates for the unknown nodal values. To obtain such estimates, the real (or imaginary) parts of $\hat{\omega}(z)$ are collocated to the m known nodal values, resulting in m equations for the m unknown nodal

values. Using these m nodal value estimates along with the m known nodal values supplies the $\hat{\omega}(z)$ integral function with sufficient data to determine the CVBEM approximation of (9).

CVBEM APPROXIMATION ERROR

Generally, numerical approximation errors in solving potential problems are of two forms: (i) errors due to not satisfying the governing equation over Ω, and (ii) errors due to not satisfying the boundary conditions continuously on Γ. For the CVBEM (and for other boundary integral equation methods), the first type of approximation error is eliminated due to both $\hat{\phi}$ and $\hat{\psi}$ being potential functions. But $\hat{\omega}(z)$ does not usually satisfy the boundary conditions continuously on Γ (if it did, then $\hat{\omega}(z) = \omega(z)$). The next step in the CVBEM analysis is to work with $\hat{\omega}(z)$ in order that $\hat{\omega}(z) \rightarrow \omega(z)$.

This step in the analysis of approximation error provides a significant advantage over domain numerical methods such as finite elements or finite differences. In the domain methods, the analyst examines error with a form of sequence Cauchy convergence criteria by arbitrarily increasing the domain nodal densities and comparing the resulting change in estimated nodal values. Whereas with the CVBEM, the analyst has several forms of the approximation error to work with.[3] Probably the easiest form of error to study is the development of the approximative boundary $\hat{\Gamma}$ which represents the locations where $\hat{\omega}(z)$ achieves the desired boundary values of $\omega(z)$. Generally, the boundary conditions are constant values of ϕ or ψ along boundary elements, i.e. $\phi = \phi_j$ for $z \in \Gamma_j$ or $\psi = \psi_k$, for $z \in \Gamma_k$. This set of m nodal values $\{\phi_j, \psi_k\}$ are level curves of $\omega(z)$. The approximative boundary $\hat{\Gamma}$ is determined by locating those points where $\hat{\phi} = \phi_j$ and $\hat{\psi} = \psi_k$ (see Fig. 1). Due to the collocation process, $\hat{\Gamma}$ intersects Γ at least at each nodal point location, $z_j, j = 1, 2, \ldots, m$.

To determine $\hat{\Gamma}$, each element Γ_j is further subdivided by interior points (specified by the program user) where $\hat{\omega}(z)$ is to be evaluated. At each element interior point, $\hat{\omega}(z)$ is calculated from the line integral of (9) and the values of $\hat{\phi}$ and $\hat{\psi}$ are determined. If the appropriate $\hat{\phi}$ (or $\hat{\psi}$) matches the boundary condition on Γ_j, then $\hat{\Gamma}$ intersects Γ at that point. Otherwise, subsequent points are evaluated by marching pointwise along a line perpendicular to Γ_j until the boundary condition value is reached. For point locations interior of Ω, equation (9) is used. For points exterior of $\Omega \cup \Gamma$, an analytic continuation of (9) is used.

In this fashion, a set of points is determined where $\hat{\omega}(z)$ equals the desired ϕ_j or ψ_k values. The contour $\hat{\Gamma}$ is estimated by then connecting these points by straight lines. Because $\hat{\Gamma}$ and Γ intersect at least at nodal point locations, $\hat{\Gamma}$ appears as a plot which oscillates about the Γ contour.

COMPUTER INTERACTION FOR ERROR REDUCTION

A procedure to use a graphical display for evaluating the CVBEM model is to display both Γ and $\hat{\Gamma}$ superimposed on the CRT. By magnification of the departure between Γ and $\hat{\Gamma}$, the analyst can easily inspect the performance of the CVBEM approximation. Because the approximation error is due to the assumed basis function assumptions, the integration error is reduced by the addition of nodal points on Γ, similar to an adaptive integration technique.

The addition of nodal points can be made directly via the CRT screen and a 'locating the closest boundary co-

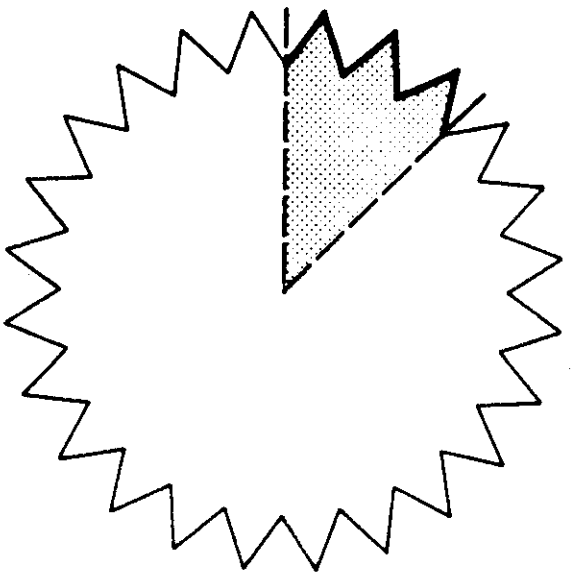

Figure 4. Example problem geometry

ordinate' computer-graphic subroutine. After the nodal additions are completed, a new $\hat{\omega}(z)$ is determined and the revised $\hat{\Gamma}$ plotted on Γ. By the addition (and deletion) of nodal points from Γ, the analyst is able to quickly evaluate the quality of the CVBEM model. Because the addition of a nodal point can be interpreted as the addition of an approximation error sink term, the geometric representation of error by means of $\hat{\Gamma}$ provides a mathematical sophisticated yet easy-to-use modeling tool.

CASE STUDY

To illustrate the previous discussion, a computer-interactive version of the CVBEM for solving potential problems in two-dimensional domains as developed by Advanced Engineering Software (Irvine, California) is considered.

The test problem considered is the development of a CVBEM approximation function for the two-dimensional domain shown in Fig. 4. This example represents any number of possible engineering problems such as listed in Table 1.

The objective of the analysis is to locate a sufficient number of CVBEM nodal points on Γ until $\hat{\Gamma}$ is within an acceptable tolerance to Γ. Generally, this tolerance is the allowable limit of deviation from the design for construction purposes.

Using symmetry, the domain of Fig. 4 is reduced to the domain of Fig. 5. The purpose of using symmetry is to reduce computational effort and computer memory requirements. Because the CVBEM is a boundary integral method, all nodal values are linked together resulting in a square matrix. Consequently the use of symmetry to reduce the problem size, or even to use the computer interaction approach rather than a brute force computer-generated nodal distribution on Γ, saves considerably on computational requirements.

Figure 6 shows the first attempt at modeling the domain of Fig. 5. Because of the nature of the approximative boundary concept, the boundary condition values of constant ϕ (or ψ) stepwise along Γ are of no real consequence. However, for the reader's convenience, the boundary conditions are also shown in Fig. 5.

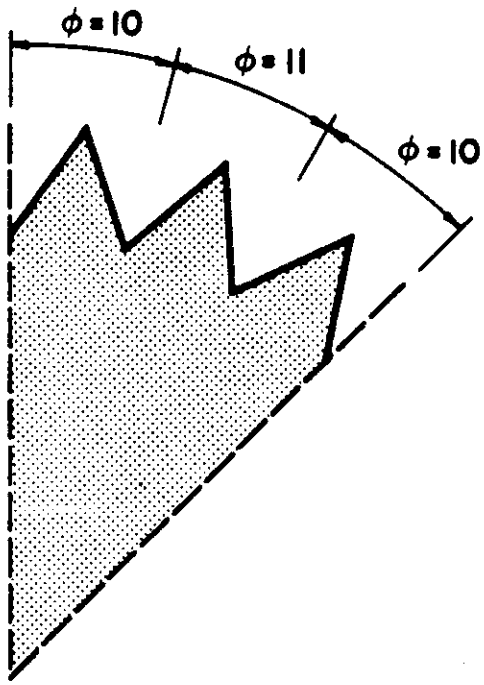

Figure 5. Simplified problem geometry

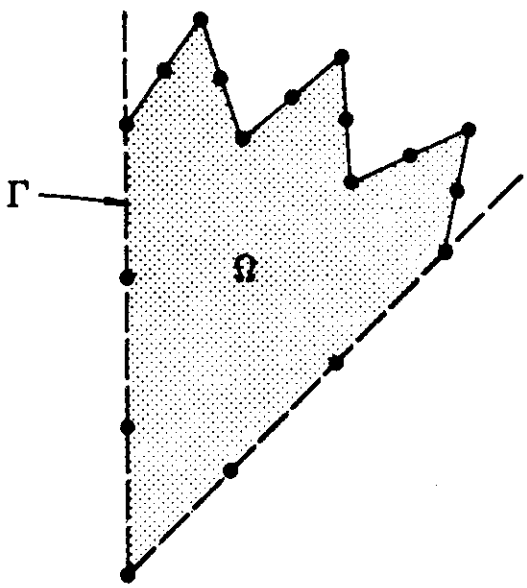

Figure 6. CVBEM nodal distribution for example problem

Figure 7 shows the overlay of Γ and $\hat{\Gamma}$ for the nodal distribution used in Fig. 6. The modeler locates additional nodes for subsequent tries based on the largest departure between Γ and $\hat{\Gamma}$. After four attempts, the CVBEM modeling error is represented by $\hat{\Gamma}$ as shown in Fig. 8. It is noted that in Fig. 8, departure is magnified ten-fold for visibility. As discussed previously, if the $\hat{\Gamma}$ is acceptable for construction purposes then the associated $\hat{\omega}(z)$ is the exact solution of the boundary value problem with Γ transformed into $\hat{\Gamma}$.

SOFTWARE PACKAGE DESIGN

Both minicomputer and microcomputer versions of the discussed CVBEM technique are available. Consequently, the software structure for an Apple II E 64K microcomputer will be presented only.

The reported CVBEM computer interaction program is subdivided into three large legs where each leg contains the main driver program.

The program package is composed of

(i) CVBEM approximation program (to determine nodal estimates)
(ii) CVBEM approximator evaluation program (to evaluate any $\hat{\omega}(z)$)
(iii) Approximative boundary determination program to determine (x, y) co-ordinates where $\hat{\omega}(z)$ equals the boundary condition level curves
(iv) line drawing graphics program to plot (x, y) pairs for both Γ and $\hat{\Gamma}$ onto CRT (or plotter)
(v) Nodal point (x, y) data entry routine

Figure 7. Approximative boundary (dashed line) for first attempt using CVBEM

Figure 8. Approximative boundary (dashed line) after four attempts using CVBEM (departures between Γ and $\hat{\Gamma}$ are magnified ten-fold)

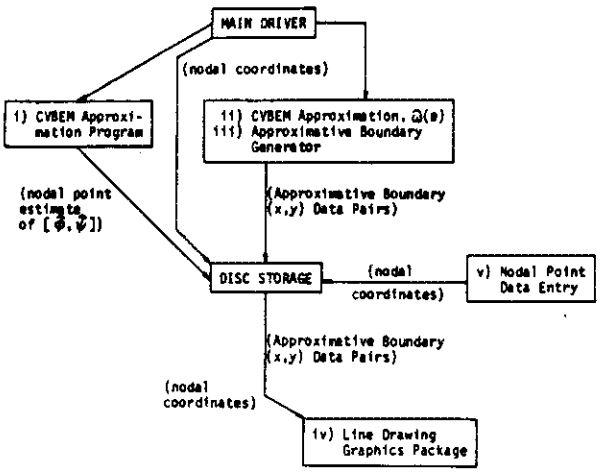

Figure 9. CVBEM computer-interaction program structure schematic

The microcomputer programming is structured as shown in Fig. 9. From the figure, disc storage is used to store $\hat{\Gamma}$ related (x, y) pairs, otherwise, computer memory is used for nodal point co-ordinates.

CONCLUSIONS

The CVBEM has been used to develop highly accurate solutions for two-dimensional potential problems. In order to achieve a high degree of accuracy, a computer interactive graphics technique is reported which utilizes the approximative boundary technique to display the CVBEM modeling error as a result of the nodal point distribution selected by the analyst. Subsequent nodal point locations can be added (or deleted) by direct interaction with the computer program via the CRT. The only programming requirements needed to implement this easy-to-use analysis approach with the CVBEM is a standard CRT line-drawing graphics package, and a 'locating a point to the closest contour' program routine.

REFERENCES

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