Locating CVBEM collocation points for steady state heat transfer problems

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The Complex Variable Boundary Element Method or CVBEM provides a highly accurate means of developing numerical solutions to steady state two-dimensional heat transfer problems. The numerical approach exactly solves the Laplace equation and satisfies the boundary conditions at specified points on the boundary by means of collocation. The accuracy of the approximation depends upon the nodal point distribution specified by the numerical analyst. In order to develop subsequent, refined approximation functions, four techniques for selecting additional collocation points are presented. The techniques are compared as to the governing theory, representation of the error of approximation on the problem boundary, the computational costs, and the ease of use by the numerical analyst.

Key Words: Laplace equation, complex variable, numerical method, error analysis.

INTRODUCTION

The numerical solution of two-dimensional steady state heat transfer problems generally is approached by use of a domain method such as finite differences or finite elements. These methods approximately solve the governing partial differential equation (Laplace equation) and the boundary conditions. Generally, the error of approximation is unavailable to the numerical analyst for use in locating additional nodal points in order to develop subsequent improved approximations.

The Complex Variable Boundary Element Method or CVBEM[1] develops an approximation function $\hat{\omega}(z)$ which exactly satisfies the two-dimensional Laplace equation. Consequently, there is no numerical error in the approximation of the governing partial differential equation. However, the CVBEM $\hat{\omega}(z)$ function generally does not satisfy the boundary conditions continuously on the problem boundary, Γ (if it did, then $\hat{\omega}(z)$ is the problem solution). Thus the approximation error appears as the discrepancy in matching the prescribed boundary conditions. Should this error be known to the numerical analyst, a refined approximator function could be subsequently developed by the means of specifying additional nodal points (or points of collocation) on Γ where the error is considered large.

In this paper, four methods for representing this approximation error are considered and compared as to computational costs, the error representations, and ease of interpretation. Through application of the techniques to steady state heat transfer problems, the considered methods' error representation are demonstrated.

CVBEM DEVELOPMENT

The CVBEM has been shown to be a useful numerical technique for the approximation of properly posed boundary value problems involving the Laplace equation.[1]

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The keystone of the numerical approach is the definition of the integral function

$$\hat{\omega}(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{G(\zeta)\, d\zeta}{\zeta - z} \tag{1}$$

where Γ is a simple closed contour enclosing a simply connected domain Ω; ζ is the variable of integration with $\zeta \in \Gamma$; z is a point in Ω; and the direction of integration is in the usual counterclockwise (positive) sense. The function $G(\zeta)$ is a global trial function which is continuous on Γ. For example, given m nodal points specified on Γ defined by co-ordinated $z_j, j = 1, 2, \ldots, m$, let $\bar{\omega}_j$ be the notation for the nodal values at node j. Then the m nodes result in m boundary element Γ_j, $j = 1, 2, \ldots, m$ where Γ_j is the straight line segment between co-ordinates z_j and z_{j+1}. A linear global trial function is defined by

$$G(\zeta) = \sum_{j=1}^{m} \delta_j (N_j \bar{\omega}_j + N_{j+1} \bar{\omega}_{j+1}) \tag{2}$$

where $\delta_j = 1$ if $\zeta \in \Gamma_j$ and $\delta_j = 0$ if $\zeta \notin \Gamma_j$. In this case, the functions N_j and N_{j+1} are the usual linear basis functions. In the above, the index situation of $j = m$ implies that index $(j + 1)$ is equal to index 1. From the definition of $G(\zeta)$ we have

$$\int_{\Gamma} \frac{G(\zeta)\, d\zeta}{\zeta - z} = \int_{U\Gamma_j} \frac{G(\zeta)\, d\zeta}{\zeta - z} = \sum_{j=1}^{m} \int_{\Gamma_j} \frac{G(\zeta)\, d\zeta}{\zeta - z} =$$

$$\sum_{j=1}^{m} \times \int_{\Gamma_j} \frac{(N_j \bar{\omega}_j + N_{j+1} \bar{\omega}_{j+1})\, d\zeta}{\zeta - z} \tag{3}$$

The CVBEM proceeds by using (3) to develop m equations as a function of the m unknowns associated with the undetermined nodal values of either $\bar{\phi}$ or $\bar{\psi}$ at each node. That is, $\bar{\omega} = \bar{\phi} + i\bar{\psi}$ where $\bar{\phi}$ and $\bar{\psi}$ are nodal values of

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the potential and stream functions respectively. Given m nodes specified on Γ_j, we necessarily know either $\bar{\phi}$ or $\bar{\psi}$ (not both) at each z_j, $j = 1, 2, \ldots, m$. Then to estimate the remaining m nodal values, $\hat{\omega}(z)$ is collected in the form of a Fredholm equation by forcing:

Class I:

$$\bar{\phi}_k(z_j) = \text{Re}\ \hat{\omega}(z_j)$$
$$\bar{\psi}_k(z_j) = \text{Im}\ \hat{\omega}(z_j) \tag{4a}$$

Class II:

$$\bar{\phi}_U(z_j) = \text{Re}\ \hat{\omega}(z_j)$$
$$\bar{\psi}_U(z_j) = \text{Im}\ \hat{\omega}(z_j) \tag{4b}$$

In the above, the subscript U and k refer to unknown and known boundary condition nodal values, respectively. Because $\hat{\omega}(z_j) = \hat{\omega}(\bar{\phi}_k, \bar{\phi}_U, \bar{\psi}_k, \bar{\psi}_U)$, then a $\hat{\omega}(z)$ is determined by either (4a) or (4b) for $j = 1, 2, \ldots, m$. The difference between these two approximations is that the class I system results in a CVBEM approximator which matches all the known nodal point boundary condition values whereas the class II system results in an approximation which equals the CVBEM-estimated unknown nodal point boundary condition values. From the above, the class I matrix system solves for a Fredholm equation of the first kind whereas the class II system solves for a Fredholm equation of the second kind.

Because $G(\zeta)$ is continuous on each Γ_j and also on the union of the Γ_j, then the integral function of (1) for $\hat{\omega}(z)$ is analytic for all $z \in \Omega$. Thus $\hat{\omega}(z)$ has complex derivatives of all orders for $z \in \Omega$ and $\hat{\omega}(z)$ can be written in terms of two harmonic conjugate functions by $\hat{\omega}(z) = \hat{\phi}(z) + i\hat{\psi}(z)$. Both the approximation $\hat{\phi}(z)$ and $\hat{\psi}(z)$ functions satisfy the Laplace equation exactly for any $z \in \Omega$.

The construction procedure continues by attempting to match the boundary conditions continuously on Γ. That is, we know values of ϕ or ψ at each nodal point z_j. Thus we also know either ϕ or ψ continuously along each Γ_j. However, the CVBEM class I approximator generally only equals the boundary conditions at nodal points whereas the class II system results in a $\hat{\omega}(z)$ which may not equal a boundary condition value at any nodal point. If $\hat{\omega}(z)$ equals the boundary conditions continuously on Γ, then $\hat{\omega}(z)$ is the exact solution to the boundary value problem.

Nodal equations are determined by taking the limit as the point $z \in \Omega$ approaches a selected nodal point z_j by

$$\hat{\omega}(z_j) = \lim_{z \to z_j} \frac{1}{2\pi i} \int_{\Gamma} \frac{G(\zeta)\, d\zeta}{\zeta - z} \tag{5}$$

The limiting value is also the Cauchy Principal Value, and by using either the class I or II systems, a set of m equations results which are solvable for the unknown nodal values by the usual matrix solution techniques such as Gaussian elimination.

COLLOCATION POINT DETERMINATION

The main purpose of this paper is to compare the effectiveness in reducing approximation error from the CVBEM by use of four error analysis techniques for the locating of additional collocation points on Γ. These techniques can be implemented as a separate computer process internal to the main CVBEM system program. A description of the techniques considered are in the following:

Method 1

A plot of relative error in matching boundary conditions continuously on Γ is obtained by subtracting the approximator function values (along Γ) from the known boundary condition values. Since only one of conjugate functions (ϕ or ψ) is known as a boundary condition at a point, this relative error plot is a representation of the mixed boundary condition fit. If the class I system is used, then further computation effort is needed due to this type of relative error being zero at nodal points. Thus, interior values of $\hat{\omega}(z)$ are computed on each Γ_j. If the class II system is used, however, this relative error is determined to be usually nonzero at nodal points, and is readily determined. After the determination of the relative error plot (in matching boundary conditions), additional collocation points are located near the points of larger error. Should the error be zero on each Γ_j, then $\hat{\omega}(z)$ satisfies the Laplace equation and also the prescribed boundary conditions, and $\hat{\omega}(z)$ is the solution to the problem.

Method 2

Generally, the prescribed boundary conditions are values of constant ϕ or ψ on each Γ_j. These values correspond to level curves of the analytic function $\omega(z) = \phi + i\psi$. After determining a $\hat{\omega}(z)$, it is convenient to determine an approximate boundary $\hat{\Gamma}$ which corresponds to the level curves of $\hat{\omega}(z) = \hat{\phi} + i\hat{\psi}$ which are specified as the prescribed boundary conditions. Use of the class I system is preferable due to $\hat{\Gamma}$ intersecting Γ at each nodal point. The resulting contour $\hat{\Gamma}$ is a visual representation of approximation error, and $\hat{\Gamma}$ coincident with Γ implies that $\hat{\omega}(z) = \omega(z)$. Additional collocation points are located at regions where $\hat{\Gamma}$ deviates substantially from Γ.

A difficulty in using this method of locating collocation points is that the contour $\hat{\Gamma}$ cannot be determined for points z outside of $\Omega \cup \Gamma$. To proceed, an analytical continuation of $\hat{\omega}(z)$ to the exterior is achieved by rewriting the integral function (1) in terms of

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{G(\zeta)\, d\zeta}{\zeta - z} = R_1(z) + \sum_{j=1}^{m} (\alpha_j + i\beta_j)(z - z_j) \ln(z - z_j) \tag{6}$$

where α_j and β_j are real numbers; and $\ln(z - z_j)$ is a principal value logarithm with branch-cuts drawn normal to Γ from each branch point z_j such as shown in Fig. 1. The resulting approximation is analytic everywhere except on each branch-cut. The $R_1(z)$ function in equation (6) is a first order reference polynomial which results due to the integration circuit of 2π radians along Γ. If $\omega(z)$ is not a first order polynomial, then $R_1(z)$ can be omitted in (6).

Implementation on a computer is direct although considerable computation effort is required. One strategy for using this technique is to subdivide each Γ_j with several internal points (about four to six) and determine $\hat{\omega}(z)$ at each point. Next, $\hat{\Gamma}$ is located by a Newton-Raphson stepping procedure in locating where $\hat{\omega}(z)$ matches the prescribed level curve. Thus, several evaluations of $\hat{\omega}(z)$ are needed to locate a single $\hat{\Gamma}$ point. The end product, however, may be considered very useful since it can be argued that $\hat{\omega}(z)$ is the exact solution to the boundary value problem with Γ transformed to $\hat{\Gamma}$, and $\hat{\Gamma}$ is a visual indication of approximation error.

Method 3

This technique includes features from both methods 1 and 2, and yet involves a significant reduction in computation effort over method 2 alone. First, the relative error

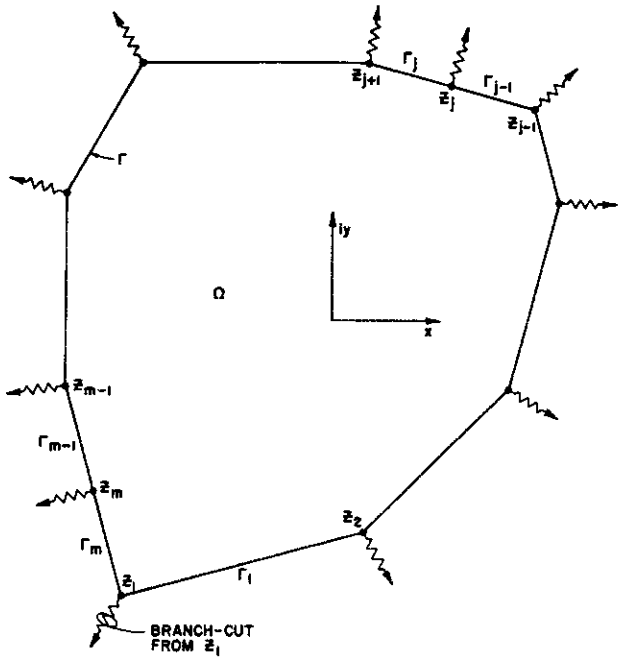

Figure 1. The analytic continuation of $\hat{\omega}(z)$ to the exterior of $\Omega \cup \Gamma$. Note branch cuts along Γ at nodes z_j

distribution of method 1 is determined along Γ between the known function (ϕ or ψ) of $\omega(z)$ and the corresponding approximation of $\hat{\omega}(z)$. The next step is to weight the relative error determined above (designated as $e(z)$ for $z \in \Gamma$) by the tangential gradient of the function conjugate to the local boundary condition variable. For example if ϕ is known on Γ_n, then for $z \in \Gamma_n$ we have $e(z) = \phi - \hat{\phi}$. This relative error is weighted by $\partial\hat{\psi}/\partial s$ which is determined directly by finite-differences of $\hat{\omega}(z)$. From the Cauchy-Rieman relations,

$$\frac{\partial\hat{\psi}}{\partial s} = \frac{\partial\hat{\phi}}{\partial n} \tag{7}$$

and an estimated distance of departure $d(\hat{\Gamma}, \Gamma, z)$ (see Fig. 2a) between the approximative boundary $\hat{\Gamma}$ and the problem boundary Γ at point $z \in \Gamma$ is given by

$$d(\hat{\Gamma}, \Gamma, z) \simeq \left| e_\phi(z) \Big/ \frac{\partial\hat{\psi}}{\partial s} \right| \tag{8}$$

In (8) the error $e_\phi(z)$ has a subscript notation that the error considered is for the ϕ function. A similar relationship holds for a specified $e_\psi(z)$ error function by

$$d(\hat{\Gamma}, \Gamma, z) \simeq \left| e_\psi(z) \Big/ \frac{\partial\hat{\phi}}{\partial s} \right|$$

The final form of error used $V(z)$ is

$$V(z) = \begin{cases} e_\phi^2(z) \Big/ \left| \dfrac{\partial\hat{\psi}}{\partial s} \right|, & \text{if } \phi \text{ is known at } z \\ e_\psi^2(z) \Big/ \left| \dfrac{\partial\hat{\phi}}{\partial s} \right|, & \text{if } \psi \text{ is known at } z \end{cases} \tag{9}$$

In the last step, additional collocation points are defined at locations where $V(z)$ is a maximum.

An advantage of method 3 over method 1 is that more weight is given to the relative error which also has a large distance of departure between Γ and $\hat{\Gamma}$. Similarly, method 3 provides an improved definition of the error associated with the approximative boundary of method 2 by including the description of whether $e(z)$ is large or small and $\hat{\Gamma}$ has a large departure from Γ simply due to a small normal gradient of the specified boundary condition variable. Figure 2 illustrates a geometric interpretation of $V(z)$ as a 'point area of error' in the CVBEM approximation. From the figure, the positive area at point z_0 equals one-half of the quantity defined in the relations given in (9). Also shown in the figure is the actual approximation value $\hat{\phi}(n)$ as a function of normal distance (n) from point $z_0 \in \Gamma$ (in the example, ϕ is the known boundary condition function).

Method 4

Because $G(\zeta)$ is continuous on Γ, $\hat{\omega}(z)$ is analytic in Ω. Thus for $z \in \Omega$ and $z \notin \Gamma$

$$\hat{\omega}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{G(\zeta)\,d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_\Gamma \frac{\hat{\omega}(\zeta)\,d\zeta}{\zeta - z} \tag{10}$$

But for $z_0 \in \Gamma$, the limit as $z \to \Gamma$ (where $z \in \Omega$) can be determined and an error $E(z_0)$ is defined by

$$E(z_0) = \lim_{z \to z_0} \frac{1}{2\pi i}\int_\Gamma \frac{G(z_0)\,d\zeta}{\zeta - z_0} - \frac{1}{2\pi i}\int_\Gamma \frac{\hat{\omega}(\zeta)\,d\zeta}{\zeta - z_0}$$

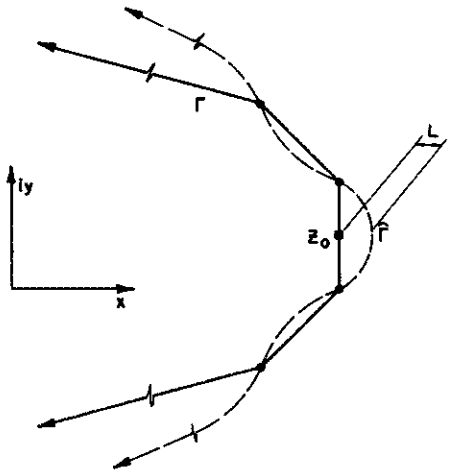

Figure 2a. Maximum departure (L) between Γ and $\hat{\Gamma}$

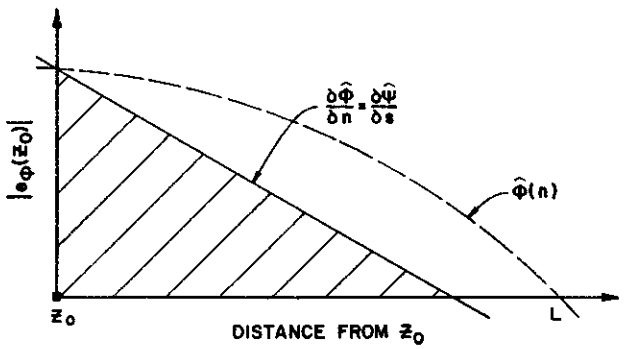

Figure 2b. Area of error at point $z_0 \in \Gamma$

or simply

$$E(z_0) = \lim_{z \to z_0} \frac{1}{2\pi i} \int_{\Gamma} \frac{[G(z_0) - \hat{\omega}(\zeta)]\, d\zeta}{\zeta - z_0} \qquad (11)$$

Collocating $E(z_j) = 0$ for $j = 1, 2, \ldots, m$ determines a class I or II system of equations (analogous to the previously discussed systems) which are used to estimate values for the unknown nodal variable function. The objective in this method is to obtain a global trial function such that in the limit $G(z_0) = \hat{\omega}(z_0)$ for all $z_0 \in \Gamma$. Thus additional collocation points are located on Γ where $|G(z_0) - \hat{\omega}(z_0)|$ is a maximum.

A comparison of method 4 to method 1 indicates that method 4 involves approximately the same computational effort as method 1, yet method 4 includes an error contribution for both the potential and stream functions. Thus a total error magnitude is provided by this technique which is not immediately available by the other three approaches.

Table 1 summarises the main features of the four considered collocation point location techniques. Included in the table are estimates of the computational effort (in CPU time) expressed as a ratio of the considered technique versus method 1. It is noted that although method 2 (approximative boundary) provides a convenient representation of the error of approximation, it requires a computational effort about 15 times at large as any of the other techniques.

APPLICATIONS

The use of the methods discussed for locating additional collocation points on Γ is demonstrated by application of the CVBEM for solving two steady state heat transfer problems. The problems considered each involve a different geometry and set of boundary conditions of the Dirichlet class. The analytic solution to the problems are included in Fig. 3. Each solution satisfies the Laplace equation and is defined as a function of a local co-ordinate $x - y$ system with an origin specified as shown in the figures. One the problem boundaries, Γ, the potential function or temperature is also a continuous function of position defined by

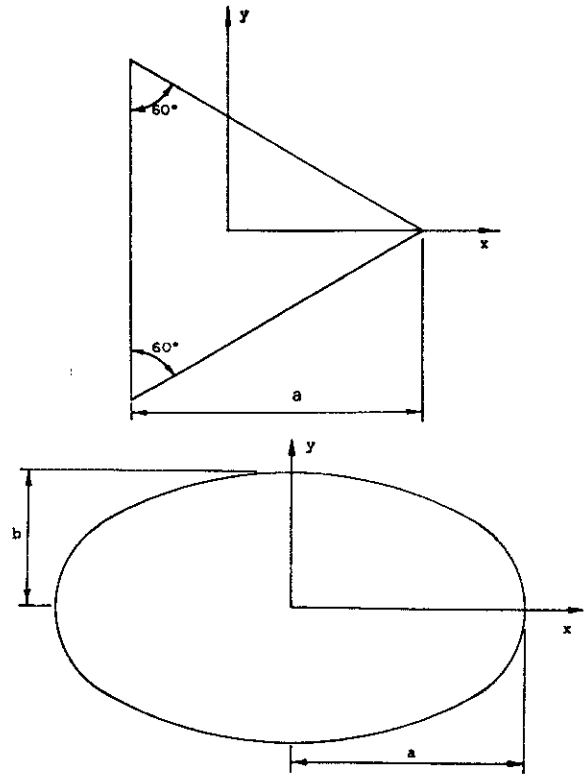

Figure 3. Application problem geometrics and exact solutions for temperature, $\phi(x, y)$. Top: (a) $\phi(x, y) = (x^3 - 3xy^2)/2a + 2a^2/27$. Below (b) $\phi(x, y) = (x^2 + y^2)/2 - a^2b^2(x^2/a^2 + y^2/b^2 - 1)/(a^2 + b^2)$

Table 1. Features of collocation point location techniques

Method number	1	2	3	4
Error analysis approach	Relative error of known boundary condition	Approximate boundary	Error area	Total relative error
Collocation point locating criteria	Maximum value of error	Maximum departure between $\hat{\Gamma}$ and Γ	Maximum point area of error	Maximum value of error
Computational effort	Single point evaluation of $\hat{\omega}(z)$	Iteration of $\hat{\omega}(z)$ for each point of $\hat{\Gamma}$	Four to six evaluations of $\hat{\omega}(z)$ for Γ_j	Single point evaluation of $\hat{\omega}(z)$
Representation of error	Relative error plot of boundary condition match	Plot of $\hat{\Gamma}$ for comparison with Γ	Plot of relative error area along Γ	Plot of total relative error $\lvert G(z) - \hat{\omega}(z) \rvert$
CVBEM class type used for estimate of unknown nodal values	II	I	II	II
Evaluation of $\hat{\omega}(z)$ at nodes	Yes	No	Yes	Yes
Evaluation of $\hat{\omega}(z)$ at points within Γ_j	No	About four to six	No	No
Includes contributions of both harmonic functions	No	No	Yes	Yes
Approximate ratio of computational effort with method 1	100%	1700%	120%	110%

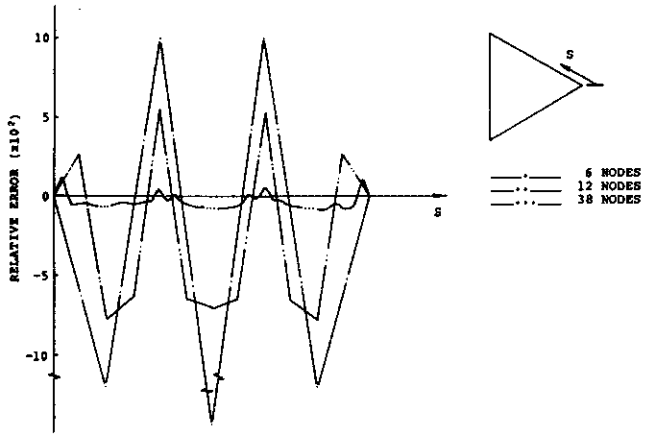

Figure 4a. Boundary relative error plot (method 1)

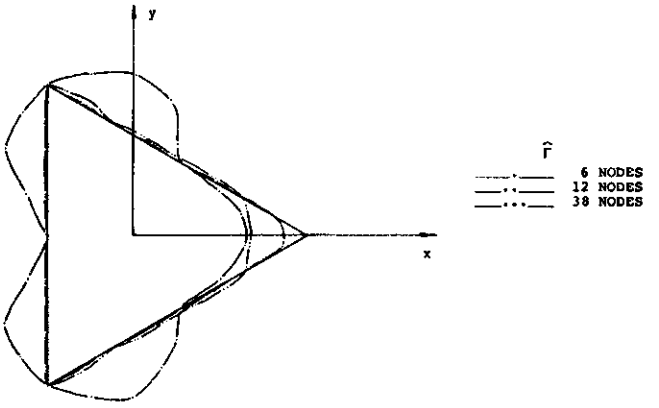

Figure 4b. Approximate boundaries for three nodal point distributions (method 3)

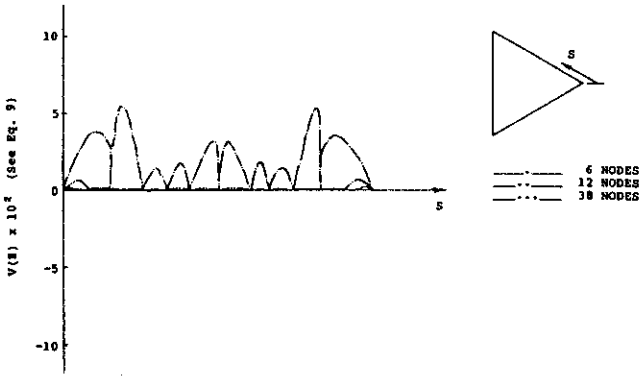

Figure 4c. Area error plot along boundary (method 2)

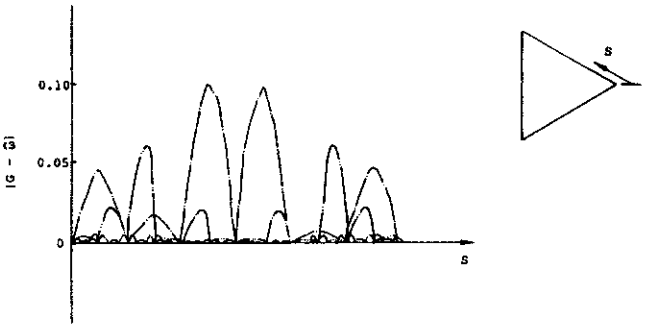

Figure 4d. Trial function error plot along boundary (method 4)

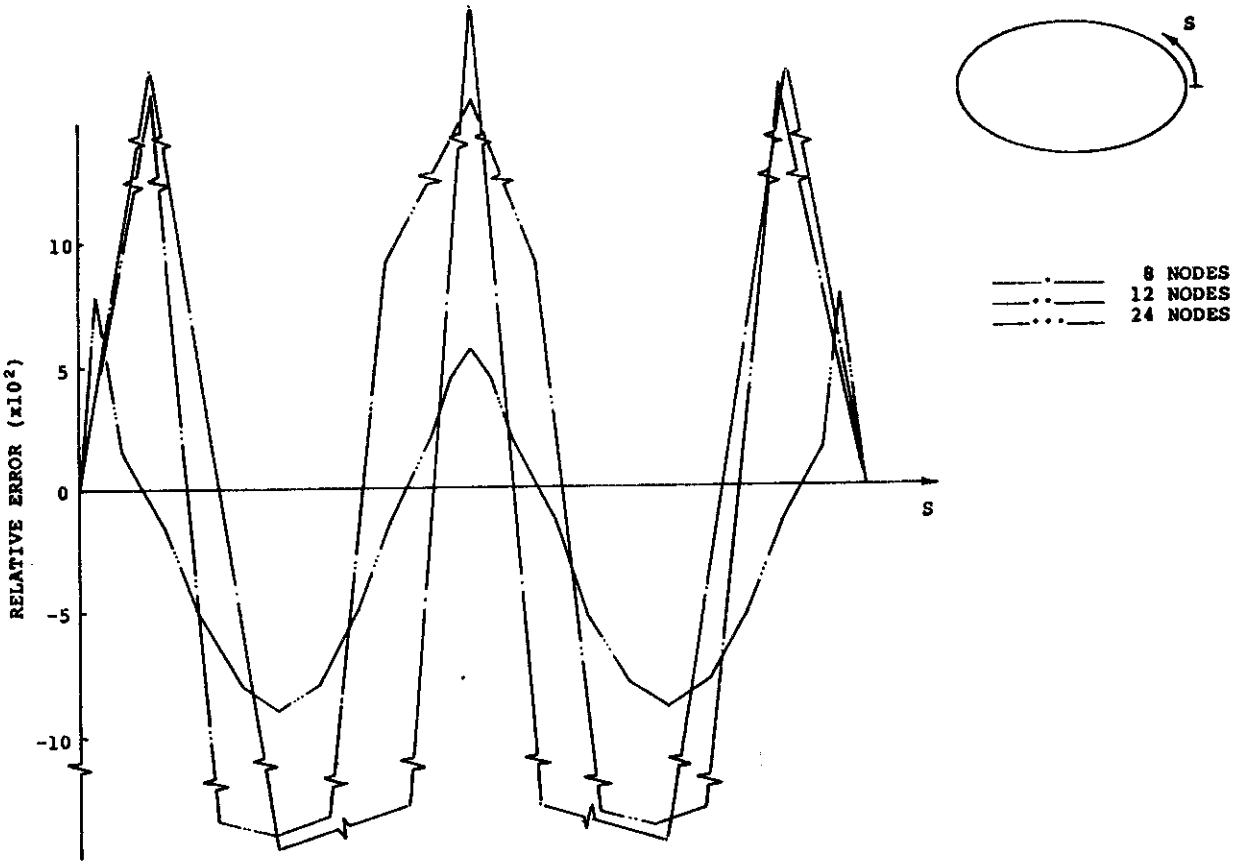

Figure 5a. Boundary relative error plot (method 1)

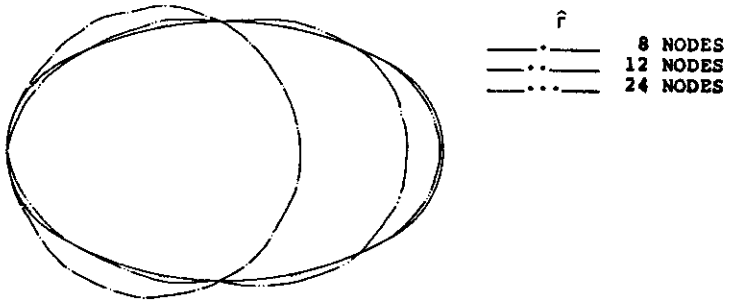

Figure 5b. Approximate boundaries for five nodal point distributions (method 3)

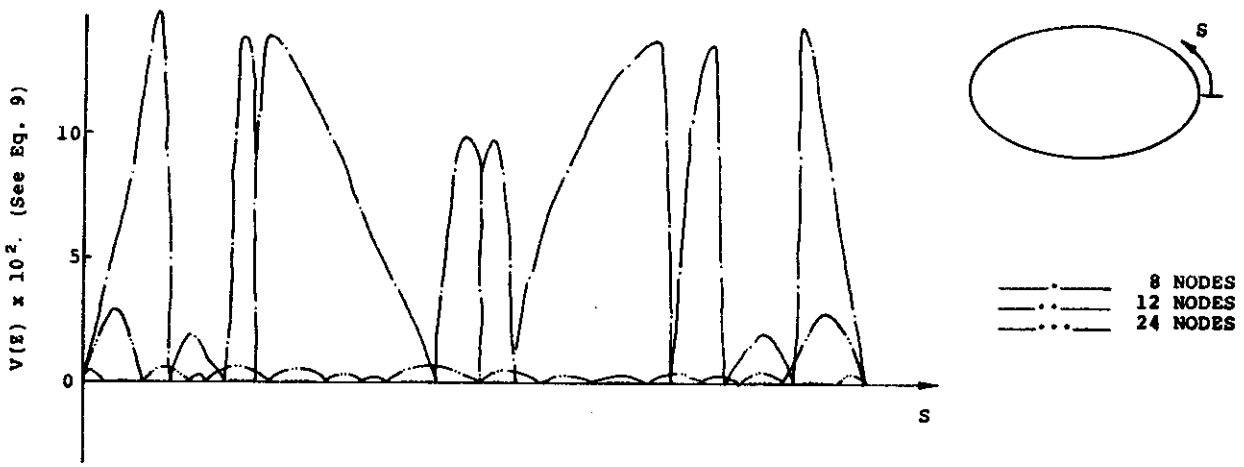

Figure 5c. Area error plot along boundary (method 2)

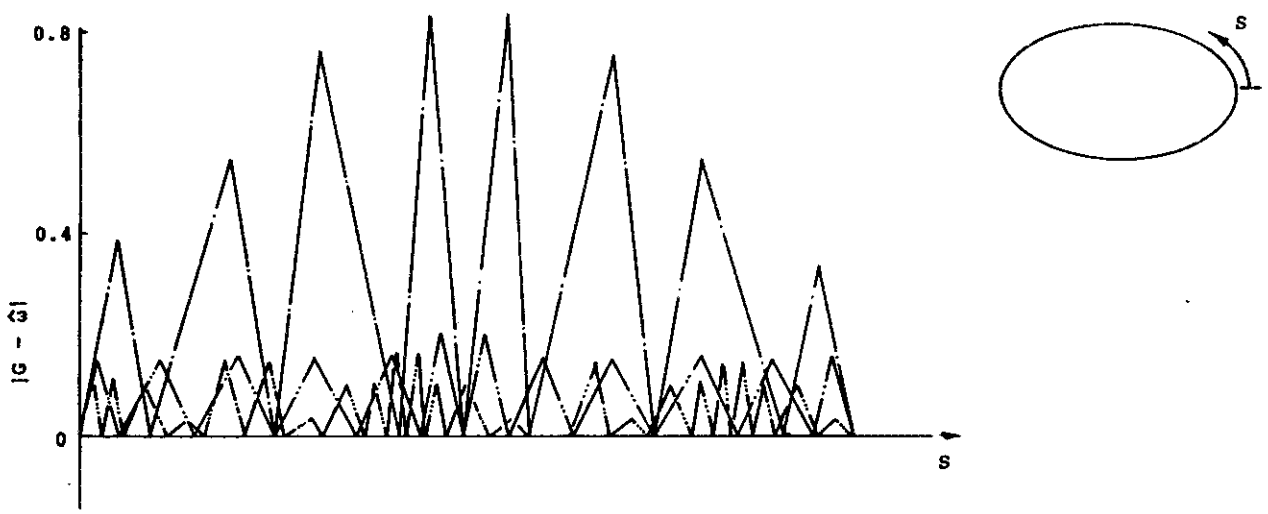

Figure 5d. Trial function error plot along boundary (method 4)

$$\phi(z \in \Gamma) = \tfrac{1}{2}(x^2 + y^2) \qquad (12)$$

From (12), it is seen that the boundary conditions are not level curves; consequently, the determination of an approximative boundary $\hat{\Gamma}$ (for method 2) requires further definition. In these applications, the problem is approached by using the statement

$$\hat{\Gamma} \equiv \{z : \hat{\phi}(z) = \tfrac{1}{2}(x^2 + y^2) = \tfrac{1}{2}|z|^2\} \qquad (13)$$

The strategy of working with level curves (i.e. $\phi = \phi_j$ for $z \in \Gamma_j, j = 1, 2, \ldots, m$) follows analogously.

The two applications illustrate the development of CVBEM approximation functions which exactly satisfy the governing partial differential equation (Laplace equation)

in Ω and approximately satisfy the boundary conditions which are continuously specified on Γ. The subsequent figures illustrate the several error evaluations along Γ for evenly spaced nodal placements for each problem boundary.

The various descriptions of error in approximating the boundary conditions all indicate regions on Γ where discrepancy is large. Additional nodal points would be specified at these locations of large deviation, and a refined CVBEM approximation function developed. This process is repeated until the analyst is satisfied with the error distribution on Γ.

From the figures, methods 1, 3 and 4 provide similar abstract representations of the error in approximation. However, method 2 results in a visual representation of approximation error which is directly interpretable by most numerical analysts. Often it can be argued that the mathematical description of the precise problem boundary Γ is not achieved due to the construction of the prototype, and that the approximate boundary $\hat{\Gamma}$ may actually represent a more probable end product. Because $\hat{\omega}(z)$ is the exact solution of the boundary value problem with Γ transformed into $\hat{\Gamma}$, then the selection of a $\hat{\Gamma}$ has the advantage of also being associated with the generating $\hat{\omega}(z)$ solution. From the Table, however, method 2 requires nearly 15 times the computational effort as the other methods considered.

DISCUSSION

From the applications, the CVBEM approach provides a very useful tool for the numerical approximation of boundary value problems of the Laplace equation such as occurs in steady state (or even quasi steady state problems such as a slow moving interface which occurs with freezing or thawing) heat transfer processes. The CVBEM also is easily tractable to various error evaluation strategies for the subsequent location of additional collocation points on the problem boundary. These techniques are especially useful in a computer-aided-design environment where the plots of approximation error along Γ or the approximate boundary $\hat{\Gamma}$ superimposed on Γ are made visible to the analyst on the computer CRT screen immediately upon determination. The computer program then continues by either prompting for the user to specify additional nodal point co-ordinates on Γ, or by internally locating an additional fixed number of nodal points based on the location of a maximum deviation.

It is noted that since the CVBEM reduces to two real variable line integrals, the real variable Boundary Element Method or BEM[2] can also utilise the presented techniques for locating additional collocation points.

CONCLUSIONS

Four methods for locating additional collocation points for use with the CVBEM applied to steady state heat transfer problems are presented. Three of the methods considered utilise (1) relative error in matching the boundary conditions, (2) the relative error area in matching the boundary conditions, or (3) the relative error between the global trial function (defined on the boundary, Γ) and the approximation function evaluated on Γ, The above methods involve a variation in computational requirements on the order of 20%. The second method considers an approximative boundary $\hat{\Gamma}$ upon which the CVBEM approximation provides an exact solution to the boundary value problem with Γ transformed to $\hat{\Gamma}$. This method is easily interpretable by a numerical analyst and is especially suited to a computer-aided-design environment. However, the computational effort is about 15 times the effort required for the other three methods.

The applications presented illustrate the utility of the CVBEM in developing highly accurate numerical solutions to steady state heat transfer problems on regular and irregular two-dimensional surfaces.

Symbol	Description
x, y	Cartesian co-ordinates (two-dimensional)
z	$z = x + iy$
i	$i^2 = -1$
$\omega(z)$	analytic solution to boundary value problem
ϕ, ψ	potential and stream functions corresponding to $\omega = \phi + i\psi$
$\hat{\omega}(z)$	CVBEM approximator
$\hat{\phi}, \hat{\psi}$	$\hat{\omega} = \hat{\phi} + i\hat{\psi}$
Ω	problem domain
Γ	problem boundary
$\hat{\Gamma}$	approximative boundary
ζ	contour integral variable
z_0	arbitrary point
z_j	nodal point (collocation point)
Γ_j	boundary element
n, s	normal and tangential co-ordinates on contour
$e(z)$	relative error function
$e_\phi(z)$	ϕ — relative error
$e_\psi(z)$	ψ — relative error

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