VOL. 22, NO. 2 WATER RESOURCES BULLETIN AMERICAN WATER RESOURCES ASSOCIATION APRIL 1986

TWO-DIMENSIONAL DIFFUSION-PROBABILISTIC MODEL OF A SLOW DAM BREAK[1]

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ABSTRACT: A two-dimensional model of a dam-break flood wave is developed by simplifying the St. Venant equations to eliminate local acceleration and inertial terms and combining the simplified equations with continuity to form a diffusion type partial differential equation. This model is cascaded with a two point probability estimate scheme to account for uncertainty in the dam break flood hydrograph and channel roughness. The development and application of the probabilistic model is the main contribution of this paper. The approach is applied to a hypothetical dam break of Long Valley Dam on the Owens River above Bishop, California.
(KEY TERMS: dam break; two-dimensional flow model; probabilistic model.)

INTRODUCTION

The solution of the dam-break flood wave problem has received increasing importance in the U.S. and elsewhere (Chen and Armbruster, 1980; Katopodes and Schamber, 1983; Katopodes and Strelkoff, 1978; Ponce and Tsivoglou, 1981; Water Resources Council, 1977). There is an increasing number of well-documented dam-break flood wave phenomena to compare with theoretical solutions (Land, 1980; Ponce and Tsivoglou, 1981).

Most but not all theoretical solutions use the method of characteristics to solve the one-dimensional continuity and St. Venant equations. Katopodes and Schamber (1983) review several one-dimensional models. Other solution techniques to shallow wave problems are reviewed by Ponce and Simons (1977). One-dimensional solutions, however, are only generally applicable to problems where the dam-break flood wave is confined to a relatively narrow valley.

Xanthopoulos and Koutitas (1976) and Katopodes and Strelkoff (1978) advance two-dimensional solutions to the dam-break flood wave problem. Katopodes and Strelkoff (1978) extend the one-dimensional characteristic equations to the two-dimensional case and develop several interesting solutions using a moving grid algorithm. Xanthopoulos and Koutitas (1976) develop a somewhat novel solution by assuming the inertial terms to be negligible. Recently Hromadka, *et al.* (1985), used a two-dimensional diffusion type partial differential equation (p.d.e.) in the dam-break flood wave problem and solved the equation using a finite element method, again ignoring inertial terms. This solution compares well with a well documented one-dimensional code which is based upon the method of characteristics.

A simple two-dimensional numerical procedure based upon a diffusion type of p.d.e. is applied to a hypothetical dam break of the dam that impounds Lake Crowley above Bishop, California. In this study, modeling uncertainty is considered and is grouped into three categories as follows:

1. Uncertainty associated with the choice of model including the numerical analog and the spatial and temporal discretization.
2. Uncertainty associated with initial conditions and primarily boundary conditions; i.e., the input hydrograph resulting from the dam break phenomena. For example, see Ponce and Tsivoglou (1981) concerning the problem of assuming an input hydrograph.
3. Uncertainty associated with specifying model parameters, primarily Manning's roughness coefficient.

The main contribution of this work is to systematically consider the combined uncertainty of the input hydrograph and roughness coefficients by cascading the deterministic diffusion type p.d.e. model with a probabilistic model based upon a point probability estimation technique developed by Guymon, *et al.* (1981). As the reader will subsequently see, this technique involves considerably less computational effort than required by the Monte Carlo technique, which requires hundreds if not thousands of simulations. Moreover, the Monte Carlo technique requires an *a priori* assumption concerning the statistical distribution of uncertain boundary contions and parameters; the point probability estimate technique does not.

A two-dimensional solution is required for the Bishop, California, problem because of the nature of the terrain. The hypothetical problem is as follows. Recent seismic activity in

[1]Paper No. 85005 of the *Water Resources Bulletin.* Discussions are open until December 1, 1986.
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the vicinity of the Mammouth Caldera have given some concern for the safety of communities downstream from Lake Crowley, which is impounded by Long Valley Dam about 15 miles east of the caldera. Long Valley Dam dams Owens River about 24 river miles above Bishop. This dam is a zoned compacted earth fill structure about 100 feet in height above the streambed. The river downstream from the dam is confined in the relatively narrow Owens River Gorge for about 16 river miles, after which the terrain opens onto relatively flat alluvial slopes that form the upper part of Owens Valley. At Bishop, the valley widens to about 12 miles where Owens River turns and flows generally south through Owens Valley, which is flanked by the Sierra Nevada on the west and the White Mountains on the east. The study area location is shown in Figure 1.

DETERMINISTIC DIFFUSION MODEL

An approximation of two-dimensional wave motion is described by continuity

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial z}{\partial t} = 0 \qquad (1)$$

and the so-called Saint Venant equations in two dimensions

$$\frac{\partial Q_x}{\partial t} + \frac{\partial Q_x^2/A_x}{\partial x} + \frac{\partial Q_x Q_y/A_x}{\partial y} + gA_x\left(\frac{\partial h}{\partial x} + S_{fx}\right) = 0$$

$$\frac{\partial Q_y}{\partial t} + \frac{\partial Q_y^2/A_y}{\partial y} + \frac{\partial Q_x Q_y/A_y}{\partial y} + gA_y\left(\frac{\partial h}{\partial y} + S_{fy}\right) = 0 \qquad (2)$$

where t = time, x and y (and the subscripts) = orthogonal directions in the horizontal plane, Q = discharge, q = discharge per unit width normal to flow directions (specified by subscript), A = cross-sectional area relative to subscripted direction, z = depth of flow, h = elevation of water surface above some arbitrary datum (i.e., h is the hydraulic grade surface), g = gravitational constant, and S_f = the friction slope where

$$S_f = \frac{v^2}{C^2 R} \qquad (3)$$

where R = hydraulic radius, and C is given by Manning's relationship for C; i.e.,

$$C = \frac{C_m}{n} R^{1/6} \qquad (4)$$

where C_m = 1.486 for British units and n = Manning's roughness coefficient. Equations (2) and (3) assume pressures obey the hydrostatic law and velocities are reasonably uniform in

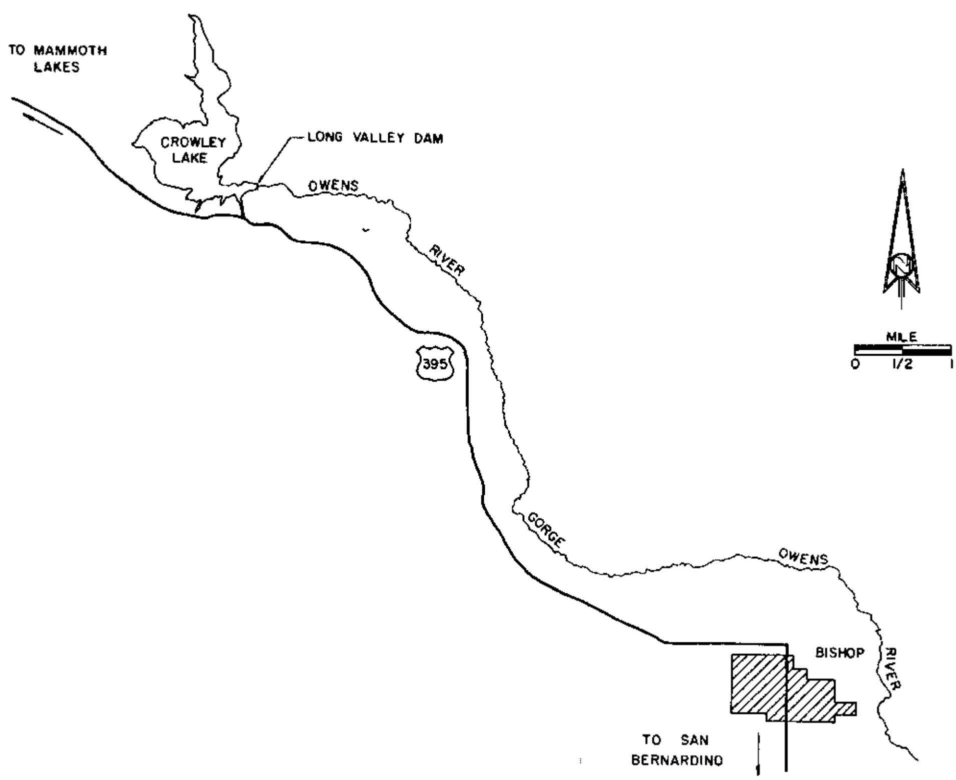

Figure 1. Study Area Near Bishop, California.

the cross-section. Both assumptions are valid for reasonably uniform and flat inverts where flows are reasonably turbulent.

Typically, Equations (1) through (4) are solved by the method of characteristics (Cunge, *et al.,* 1983; Katopodes and Schamber, 1983) which can accommodate flows when the inertial terms are significant. An alternative method based upon treating the equations as a parabolic partially differential system, proposed by Xanthopoulous and Koutitas (1976) *simplifies the problem computationally.* Such a diffusion model may reasonably hold for Froude numbers less than 2, which results in positive waves being more dispersive (Henderson, 1971). Of course, negative waves already behave in a dispersive manner. When channels are reasonably steep and there are no backwater effects, inertial terms may be neglected in many instances (Cunge, *et al.,* 1983).

To arrive at a single governing parabolic partial differential equation the momentum equations, Equation (2) can be simplified by letting

$$M_x = \frac{1}{gA_x}\left[\frac{\partial Q_x}{\partial t} + \frac{\partial Q_x^2/A_x}{\partial x} + \frac{\partial Q_xQ_y/A_x}{\partial x}\right]$$

$$M_y = \frac{1}{gA_y}\left[\frac{\partial Q_y}{\partial t} + \frac{\partial Q_y^2/A_y}{\partial y} + \frac{\partial Q_xQ_y/A_y}{\partial y}\right] \tag{5}$$

Thus, Equation (2) becomes

$$M_x + \frac{\partial h}{\partial x} = -S_{fx}$$

$$M_y + \frac{\partial h}{\partial y} = -S_{fy} \tag{6}$$

where S_f can be calculated from Equations (3) and (4), i.e.,

$$Q_x = \left[\frac{C_m A_x R_x^{2/3}}{n\,|S_{fx}|^{1/2}}\right] S_{fx} \equiv K_x S_{fx}$$

$$Q_y = \left[\frac{C_m A_y R_y^{2/3}}{n\,|S_{fy}|^{1/2}}\right] S_{fy} \equiv K_y S_{fy} \tag{7}$$

where K is regarded as a channel flow conduction parameter. Equations (6) and (7) may be combined to form

$$Q_x = -K_x \frac{\partial h}{\partial x} - K_x M_x$$

$$Q_y = -K_y \frac{\partial h}{\partial y} - K_y M_y \tag{8}$$

Now combining Equation (8) with continuity, Equation (1), a single p.d.e. is obtained as follows:

$$\frac{\partial}{\partial x}\left[K_x \frac{\partial h}{\partial x}\right] + \frac{\partial}{\partial y}\left[K_y \frac{\partial h}{\partial y}\right] + S = B\frac{\partial h}{\partial t} \tag{9}$$

where $S = K_yM_y$ and B = grid width. Note that K_x and K_y are also functions of M_x and M_y.

Comparisons of approximate numerical solutions of Equation (9) in one-dimensional form to other standard dam break solutions, shows that in the type of problem considered herein, the M_x and M_y functions may be set to zero (Hromadka, *et al.,* 1985). Thus Equation (9) may be simplified to

$$\frac{\partial}{\partial x}\left[K_x \frac{\partial h}{\partial x}\right] + \frac{\partial}{\partial y}\left[K_y \frac{\partial h}{\partial y}\right] = B\frac{\partial h}{\partial t} \tag{10}$$

where $\partial h/\partial t = \partial y/\partial t$ and from Equations (6) and (7) (letting $M_x = M_y = 0$)

$$K_x = \frac{C_m A_x R_x^{2/3}}{n\left|\frac{\partial h}{\partial x}\right|^{1/2}}$$

$$K_y = \frac{C_m A_y R_y^{2/3}}{n\left|\frac{\partial h}{\partial y}\right|^{1/2}} \tag{11}$$

Equation (10) is a nonlinear p.d.e. that may be solved by using numerical techniques such as finite differences or finite elements with an appropriate iteration scheme to handle the nonlinear K_x and K_y parameters. A rather simple approach leading to a minimum of computer code is to use integrated finite differences which yields the difference equation

$$Q_x - Q_{x+\Delta x} + Q_y - Q_{y+\Delta y} = A_{xy}\frac{\Delta h}{\Delta t} \tag{12}$$

where the solution domain is discretized into square grids such that $\Delta x = \Delta y$ and A_{xy} = grid area (a constant for the solution domain). The computational procedure consists of calculating the discharge crossing each face of a grid from Equation (8) where $M_x = M_y = 0$. The coefficients, K_x and K_y, are computed from knowledge of h at time t, and similarly the gradient in Equation (8) may be computed from estimated h at time t. The change in water surface elevation, Δh, is computed from Equation (12) given a specified Δt which must be very small for this simple procedure to be stable; i.e., Δt is on the order of 0.001 minutes. Initial conditions for h are assumed zero. Boundary conditions consist of zero flow at the edges of the domain and the dam break discharge at the points where dam break discharge enters the solution domain.

PROBABILISTIC MODEL

Guymon, *et al.* (1981), develop a general point probability estimation technique as an efficient method of accounting for parameter and boundary condition uncertainty associated with a complex porous media flow system. The same technique may advantageously be applied to the dam-break flood wave problem. In fact, the method can be applied to any uncertainty problem as an alternate to the commonly used Monte Carlo method. The derivation of the point probability estimate technique will only be briefly reviewed here.

In the previous section an approximate deterministic model was developed to determine flood wave height. For simplicity this model is represented by

$$h = h(p_1, p_2, \ldots) \tag{13}$$

where h is a function of parameters and auxiliary conditions p_i. Suppose the mean value and the coefficient of variation of the p_i can be measured or inferred for M discrete values of the p_i. The statistical properties of the p_i are estimated in the usual way, where the first two moments, the expectancy and the variance, are respectively

$$E(p_i) = \overline{p}_i = \frac{1}{M} \sum_{0}^{M} (p_i)_j$$

$$V_{p_i} = E(p_i^2) - [E(p_i)]^2 = \frac{1}{M} \sum_{0}^{M} [(p_i)_j - \overline{p}_i]^2 \tag{14}$$

The coefficient of variation is given by

$$CV_i = s_{p_i}/\overline{p}_i \tag{15}$$

Where s_{p_i} is the standard deviation which is equal to the positive square root of the variance, V_{p_i}.

We seek the mean of h and its standard deviation; i.e., we seek $\overline{h}$ and s_h as an example of the use of the probabilistic technique. Alternatively, we could deal with velocities or discharge.

Guymon, *et al.* (1981), show that the N-th moment of h may be computed from the general relationship

$$E[(h)^N] = \frac{1}{2^r} [(h_{++..r})^N + (h_{-++..r})^N + \ldots + (h_{---..r})^N] \tag{16}$$

where there are r parameters and auxiliary conditions to be onsidered and the notation $h_{++..r}$ indicates all sign permutations of

$$h = h(\overline{p}_1 \pm s_{p_1}, \overline{p}_2 \pm s_{p_2}, \ldots, \overline{p}_r \pm s_{p_1}) \tag{17}$$

Equation (16) can be derived by expanding $h = h(p_i)$ in Taylor's series (Guymon, *et al.*, 1981). The mean and variance of h can be found from the first and second moments computed from Equation (16) as follows:

$$\overline{h} = E(h) \tag{18}$$

$$V_h = E[(h)^2] - [E(h)]^2 \tag{19}$$

In this particular application it is assumed that the p_i are not correlated. Guymon, *et al.* (1981), show that Equation (16) may be extended to incorporate correlation of the p_i using the convariance statistic.

The point probability estimate technique is a powerful analysis tool. Prior knowledge of the probability distribution of the p_i are not required; only the mean and coefficient of variation are required of the p_i. Only w^r simulations using the deterministic model are required as contrasted to the hundreds or thousands of simulations required by the Monte Carlo method. Moreover, the Monte Carlo method requires some knowledge of the probability distribution of the p_i.

The point probability estimate technique may be extended to compute confidence limits for the $\overline{h}$. Suppose we know nothing of the probability distribution of h, then Chebeshev's inequality may be used; i.e.,

$$P[\overline{h} - k S_h \leqslant h \leqslant \overline{h} + k S_h] \geqslant 1 - \frac{1}{k^2} \tag{20}$$

For example, if three standard deviations are used, the probability that the computed h is bounded by $\pm 3S_h$ is 89 percent. This confidence would increase if we assumed h is symmetrically distributed since Gauss' unequality applies

$$P[\overline{h} - k S_h \leqslant h \leqslant \overline{h} + k S_h] \geqslant 1 - \frac{4}{9k^2} \tag{21}$$

Using the example of $\pm 3S_h$, there would be a 95 percent confidence if Gauss' inequality applied. Assuming complete knowledge of the distribution of h may narrow the confidence limits substantially; e.g., one might assume a beta distribution which has the advantage of an infinite variety of shapes and includes the normal distribution of a subset.

APPLICATION

The deterministic-probabilistic model was applied to a hypothetical break of Long Valley Dam by assuming that the dam-break flood wave would move down Owens River Gorge and discharge into the alluvial plain northwest of Bishop, California. To solve the diffusion type model, Equation (12), the assumed flood plain was divided into a uniform grid system of connected squares one-half mile on a side. Figure 2 shows the grid layout. Each grid is represented by a node in its center where land surface elevation and roughness coefficients are specified. Computed water surface elevations are for each mode. The model was prepared so that mean cross-sectional velocities and Froude numbers are estimated for each node.

For convenience, Manning's n-values were assumed to be uniform over the solution domain area. There is no difficulty in specifying a variable n-value and assuming sectional variation of this value for purposes of the probability model. For this problem, it was assumed that the mean n-value was 0.040 and that the coefficient of variation was 50 percent. The magnitude of the assumed n-value significantly effects the speed of the flood wave and to some extent influences the width of the flood plain simulated.

The input dam break discharge hydrograph was assumed to be triangular in shape, as shown in Figure 3. As pointed out

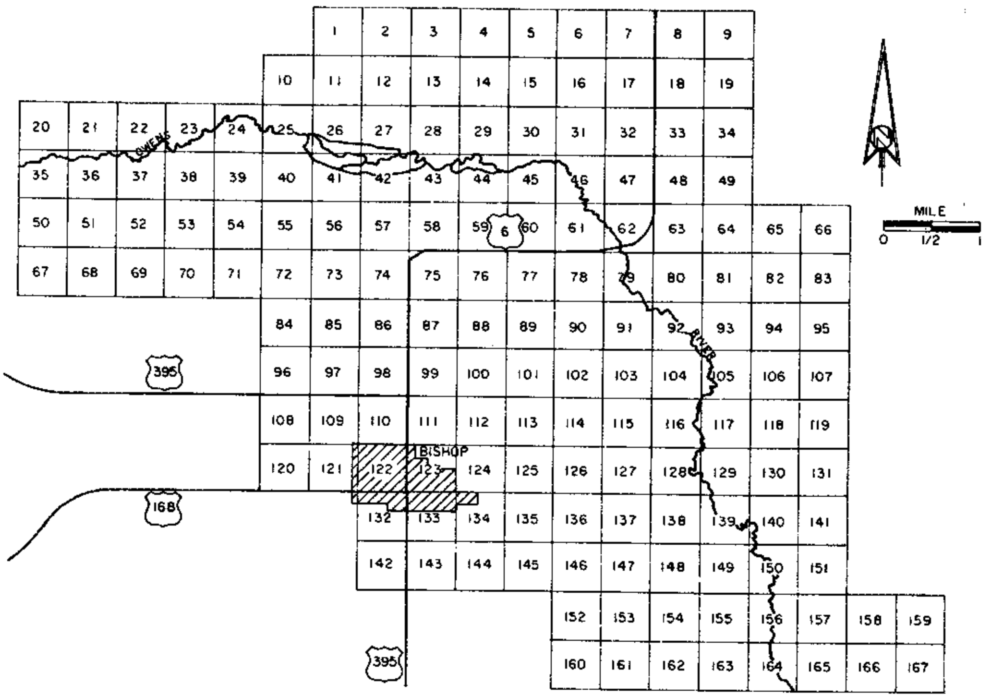

Figure 2. Grid Layout for Finite-Difference Solution

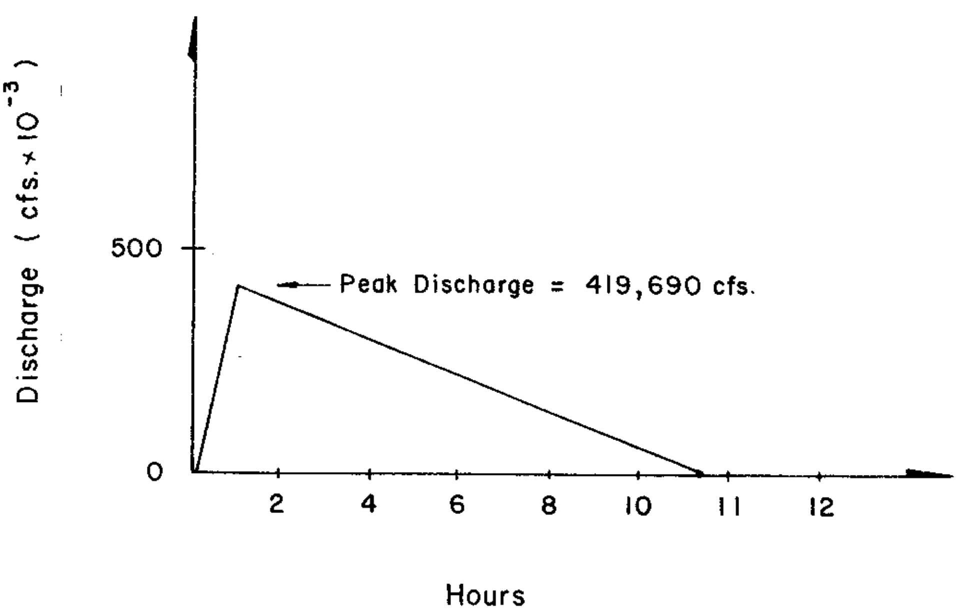

Figure 3. Assumed Dam Break Hydrograph at Node 35.

previously one of the most uncertain aspects of flood routing of dam break floods is the shape and peak discharge of the hydrograph. For the problem considered here it is assumed that the hydrograph depicted in Figure 3 is the mean condition and that the peak discharge can vary by a coefficient of variation of 30 percent. It was also assumed that the time to reach a peak (the rising limb time) may vary by a coefficient of variation of 20 percent. Principal parameters and their assumed coefficient of variation are summarized in Table 1.

TABLE 1. Assumed Parameters and Parameter Coefficient of Variation.

Parameter	Assumed Mean Value	Assumed Coefficient of Variation in Percent
Manning's-n	0.040	50
Peak Discharge	419,688 cfs	30
Rising Limb Time	60.0 min	20

The deterministic model discussed previously was used in the point probability estimate model assuming three uncorrelated parameters are of significance: roughness, dam break flood peak, and time to reach peak discharge. Only eight simulations were required to determine solution statistical means and other moments. As an example, the mean and variance of the maximum computed water depth were solved for. Other variables, such as velocity, could also be included in this scheme as well as other moments, such as skew. Results are shown in Figures 4 through 7.

Figure 4 shows the maximum flood plain inundation. The solid line is the mean and the dashed lines show plus-one and minus-one standard deviation. Assuming Gauss' inequality applies, it is estimated that the confidence that the flood plain lies within the dashed lines is 56 percent. The computed flood plain boundaries are slightly skewed south of the main river because of the way the grid was drawn. Each grid center point represents a computation point. This minor problem illustrates the need to more carefully lay out a computational grid.

Figure 5 shows a cross-section of the maximum water depths drawn northwest to southwest, roughly through the southern edge of Bishop. The computed mean and plus- and minus-one and three standard deviation lines are also depicted. Again if Gauss' inequality applies, the confidence that the maximum water depth is between plus and minus three standard deviations is 95 percent. It should be noted that Figure 5 does not depict water surface profiles at a specific time but rather shows the maximum water surface levels which occur at various times. In particular, the water surface on the right bank near Bishop, occur at a later time than those in the main floodway of Bishop River.

As can be seen from Figures 4 and 5, a rather wide range of roughness coefficient and input dam break flood hydrograph does not significantly affect the extent of the flood plain. Varying these factors do, however, substantially affect the speed of travel of the flood wave and local discharge per

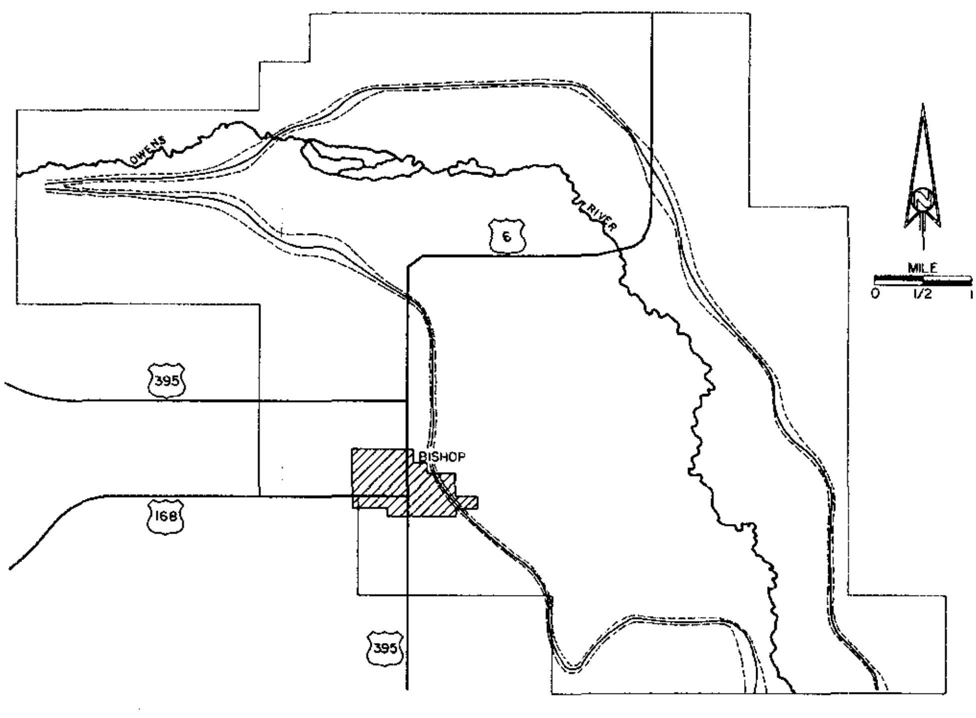

Figure 4. Dam Break Flood Plain (solid line = mean; dashed line = plus odmiries one standard deviation).

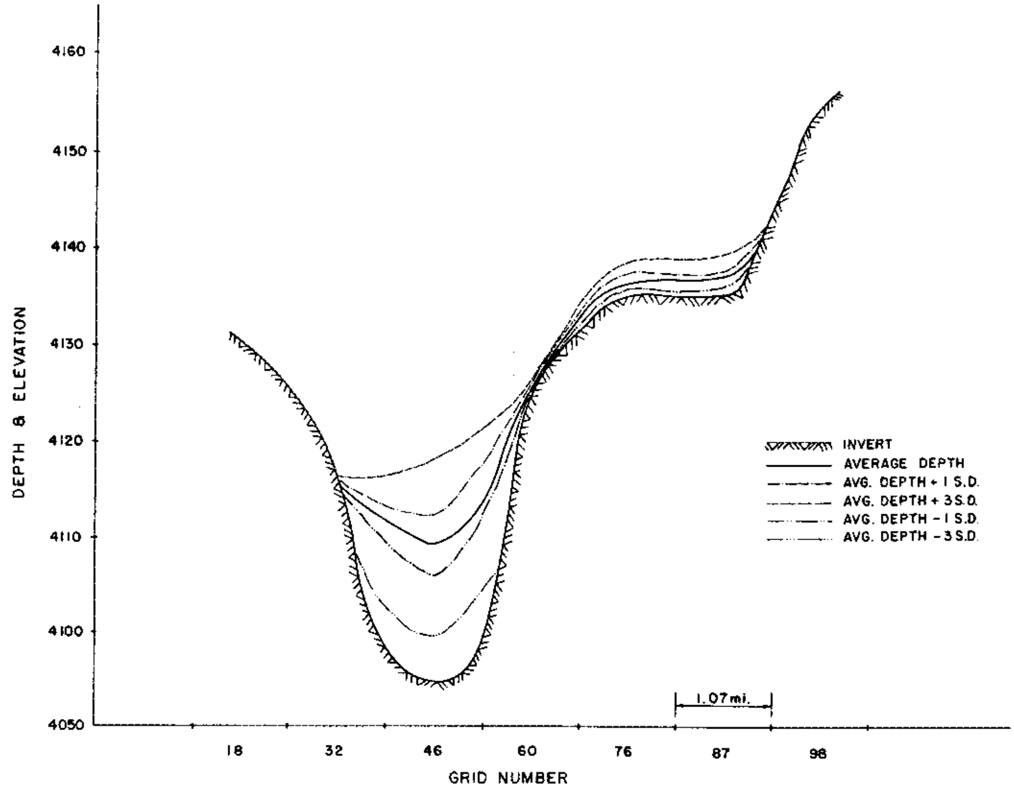

Figure 5. Cross-Section of Maximum Flood Depths Near Bishop.

unit width values. These discharge per unit width values are a measure of the destructive nature or danger of the flood, which is not only a function of depth but of velocity as well. Figure 6 depicts the extent of the dam break flood wave assuming various values for Manning's-n, inflow peak discharge, and hydrograph rising limb time. As can be seen, there is a considerable variation in the distance traveled by flood waves.

The two-dimensional nature of the problems depicted by Figure 7 which shows velocity vectors for a flood wave at 3 hours where the n-value is assumed to be 0.060, peak inflow assumed to be 293,782 cfs, and rising limb time is assumed to be 72 minutes.

The diffusion model is probably much better where water depths are larger, and has a larger relative error where computed water depths are very small. In these cases, however, the absolute error is only in fractions of a foot and are in areas of little overall importance in assessing the actual flood danger. To illustrate this point further, maximum average computer water depth ranges and their average coefficient of variation in these ranges are tabulated in Table 2. That is, this table shows a summary of the results from the point probability estimate technique. This table suggests that parameter variability affects the shallower water depths more and also the possibility that the deterministic diffusion model used is less accurate for shallower water depths.

TABLE 2. Comparison of Computed Maximum Depth Ranges and Average Coefficient of Variation.

Range of Maximum Depth in Feet	Number of Grids	Average Coefficient of Variation for Indicated Depth Range in Percent
0.001 - 0.5	31	126
0.05 - 1.0	10	94
1.0 - 2.0	10	82
2.0 - 5.0	15	50
5.0 - 10.0	22	35
10.0 - 15.0	18	29

CONCLUSIONS

The deterministic two-dimensional diffusion model of a flood wave is an approximation of an actual flood wave. However, the simplifications to arrive at this model, mainly neglecting inertial and local acceleration effects, may be warranted in some cases where two-dimensional flow effect are important. In view of the uncertainty associated with the input dam-break flood hydrograph and channel roughness coefficients, a more sophisticated solution may be of limited value. Most of these uncertainties can be simply dealt with using a two-point probability scheme that was presented herein. This approach is a powerful tool in assessing confidence limits of solutions where parameter means can be reasonably assumed

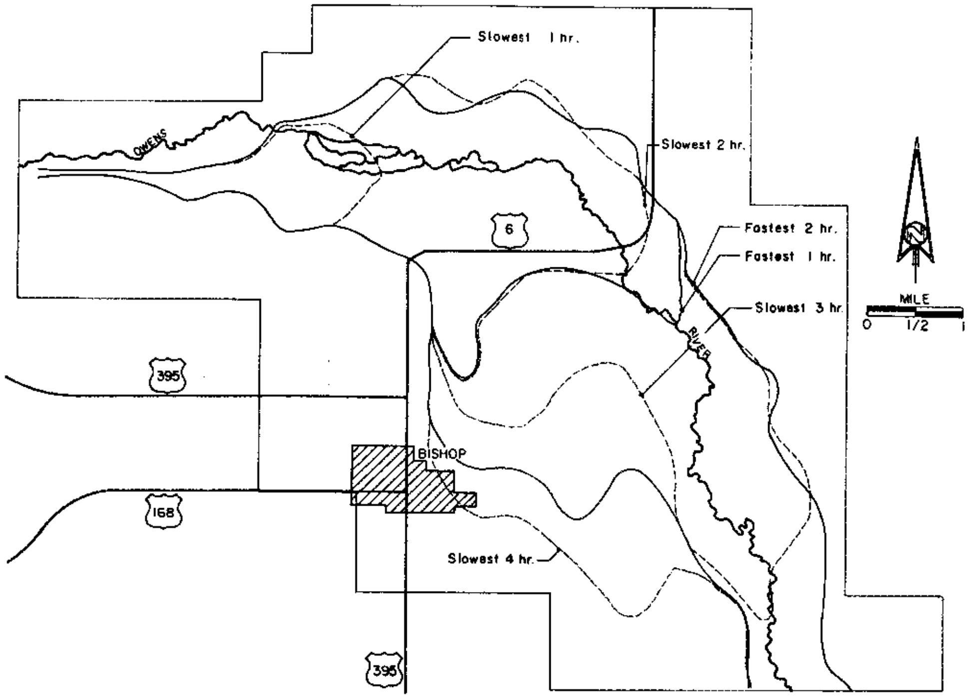

Figure 6. Maximum Extent of Flood Wave Travel at Indicated Times for Various Values of Manning's-n Value.

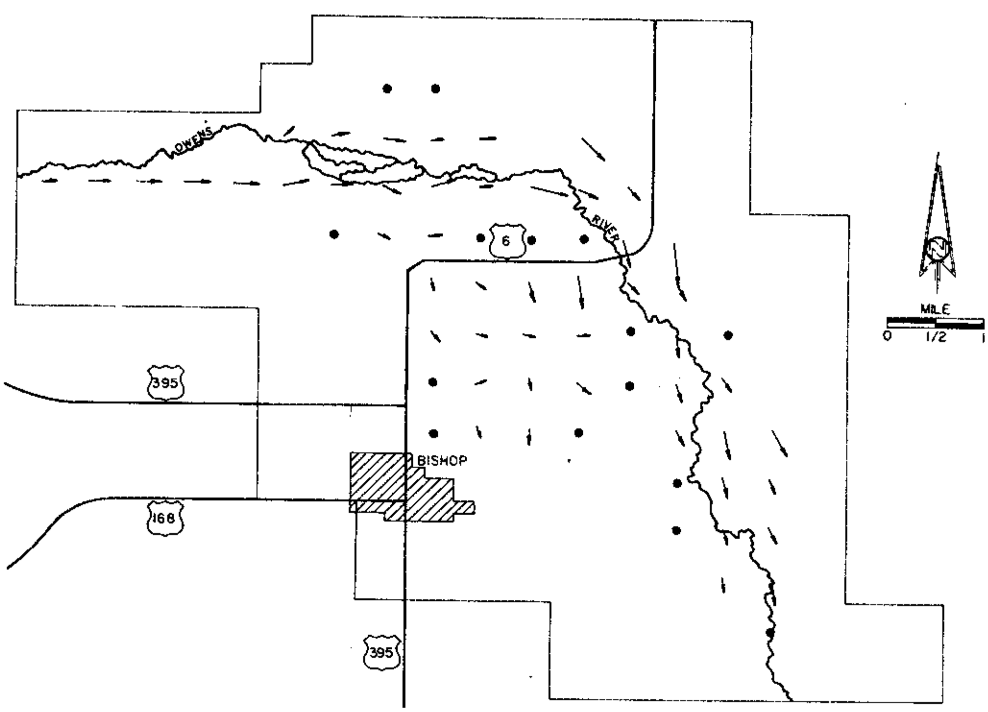

Figure 7. Flood Wave Velocity Vectors at 3 Hours for n = 0.06 and Q_{peak} = 293,782 cfs.

and their coefficient of variation estimated, or even guessed at, as was the case here.

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