$$X_0 = -\Psi - \frac{2A}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} Im\, e^{\frac{in\pi}{A}[\Psi + i(1-\Phi)]}$$

$$+ \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\sinh \frac{n\pi}{A}(1-\Phi)}{e^{\frac{2n\pi}{A}} - 1} \sin \frac{n\pi\Psi}{A} \quad \text{(A3)}$$

Let $\xi = e^{i(\pi/A)[\Psi + i(1-\Phi)]}$ so that,

$$\sum_{n=1}^{\infty} \frac{1}{n} Im\, e^{\frac{in\pi}{A}[\Psi + i(1-\Phi)]}$$

$$= Im \sum_{n=1}^{\infty} \frac{\xi^n}{n} = Im[-\ln(1-\xi)]$$

$$= -\tan^{-1}\left[\frac{-e^{-\frac{\pi}{A}(1-\Phi)} \sin(\pi\Psi/A)}{1 - e^{-\frac{\pi}{A}(1-\Phi)} \cos(\pi\Psi/A)}\right]$$

This is substituted for the first series in equation (A3) to obtain the form for X_0 in equation (24*a*) which converges rapidly. For $A = 1$ only about three terms are required; for $A = 6$, about 15 terms were used.

Predicting Two-Dimensional Steady-State Soil Freezing Fronts Using the CVBEM

T. V. Hromadka II[1]

Nomenclature

K_f, K_t = frozen and thawed conductivities
ϕ_f, ϕ_t = frozen and thawed temperatures
C = freezing front in domain Ω
Ω_f, Ω_t = frozen and thawed domains
ω = exact solution (analytic)
ϕ, ψ = function $\omega = \phi + i\psi,\ i = \sqrt{-1}$
S, n = tangential and normal coordinates
$\hat{\omega}_f, \hat{\omega}_t$ = frozen and thawed region CVBEM approximations

Introduction

In previous papers, Hromadka and Guymon [1] applied the complex variable boundary element method (CVBEM) to the problem of predicting freezing fronts in two-dimensional soil systems. Hromadka et al. [2] subsequently compare the CVBEM solution to a domain solution method and prototype data for the Deadhorse Airport runway at Prudhoe Bay, Alaska. In another work, the model is further extended to include an approximation of soil water flow [3]. An example in using domain methods to model a moving interface in heat transfer problems is given in Yoo and Rubinsky [4].

An example in the use of real variable boundary element methods [5] in the approximation of such moving boundary phase change problems and a review of the pertinent literature is given in O'Niell [6]. It is noted that although the CVBEM moving boundary phase change model of Hromadka and Guymon [1] will determine the steady-state location of the freezing front, the modeling process must evolve through a time history of the freezing front movement from a prescribed initial location. In contrast, this paper presents an alternative approach which directly focuses upon the steady-state freezing front location.

Consequently for problems where such steady-state freezing front locations are needed, the technique presented herein will significantly reduce the computational effort over that required in solving a time evolution problem.

Hromadka and Guymon [7] develop a relative error estimation scheme which exactly evaluates the relative error distribution on the problem boundary that results from the CVBEM approximator matching the known boundary conditions. This relative error determination is used to add or delete boundary nodes to improve accuracy. Thus, the CVBEM permits a direct and immediate determination of the approximation error involved in solution of an assumed Laplacian system.

The CVBEM is used instead of a real variable boundary element method due to the available modeling error evaluation techniques developed. The modeling accuracy is evaluated by the model-user in the determination of an approximative boundary upon which the CVBEM provides an exact solution. Although inhomogeneity (and anisotropy) can be included in the CVBEM model, the resulting fully populated matrix system quickly becomes large. Therefore in this paper, the domain is assumed homogeneous and isotropic except for differences in frozen and thawed conduction parameters on either side of the freezing front. The example problems presented were obtained by use of a popular 64K microcomputer (the current version of the program used in this study has the capacity to accommodate 30 nodal points).

Governing Equations and Assumptions

For steady-state conditions, the governing heat flow equations reduce to the Laplace equation (the transient heat capacitance term is omitted). The following assumptions are utilized (Fig. 1):

1 The two-dimensional soil system is rigid with negligible deformations due to frost heave. (Deformations could easily be included in a general purpose model by including an appropriate frost heave approximation procedure.)

2 The soil system is completely frozen above the freezing front and completely thawed below the freezing front (for the considered examples, frozen and thawed conductivities have a ratio of less than 10).

3 The soil–water flow is assumed negligible.

4 All boundary conditions are assumed constant for all time (mixed boundary conditions are currently neglected).

5 The soil system is homogeneous and isotropic (or the system is rescaled such that the modified domain is homogeneous and isotropic).

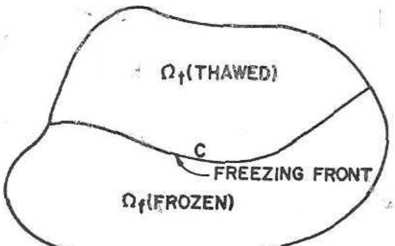

Fig. 1 Problem definition

[1]U.S. Geological Survey, Laguna Niguel, CA

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6 The effects of ice-lensing at the freezing front are ignored.

7 The steady-state heat conduction processes are modeled by the two-dimensional Laplace equation.

8 There is a single freezing front interior of the domain.

The steady-state conditions are evaluated by solving simultaneously

$$K_f \nabla^2 \phi_f = 0, \quad \text{in } \Omega_f$$
$$K_t \nabla^2 \phi_t = 0, \quad \text{in } \Omega_t \qquad (1)$$

where ϕ is the potential temperature function and (K_f, K_t) are the frozen and thawed thermal conductivities corresponding to the respective domains (Ω_f, Ω_t). On the freezing front (assumed 0°C isotherm) the conditions required are

$$\phi_f = \phi_t = 0, \quad (x, y) \epsilon C \qquad (2)$$

and for $\partial\phi/\partial n = \partial\psi/\partial S$ by the Cauchy-Riemann relationship

$$K_f \frac{d\psi}{dS} = -K_t \frac{d\psi}{dS} \qquad (3)$$

where (ϕ_f, ϕ_t) are the frozen and thawed temperatures on the freezing front contour C; ψ is the stream function which is conjugate to the potential ϕ; and S is a tangential coordinate on C, in a counterclockwise direction.

Modeling Approach (Simultaneous Potential Problem Solutions)

The modeling approach initiates by developing a CVBEM approximator $\hat{\omega}_f(z)$ and $\hat{\omega}_t(z)$ for the frozen and thawed domains, respectively. Hromadka and Guymon [8] give the details for developing such CVBEM approximators. The numerical technique determines the analytic function $\hat{\omega}(z)$ which satisfies the boundary conditions of either normal flux or temperature specified at nodal points located on the problem boundary Γ. Because $\hat{\omega}(z)$ is analytic throughout the interior domain Ω which is enclosed by Γ, then the real and imaginary parts of $\hat{\omega}(z) = \hat{\phi}(z) + i\hat{\psi}(z)$ both exactly satisfy the Laplace equation over Ω. (This property afforded by the CVBEM is not guaranteed by any of the domain methods such as finite elements on finite differences.)

For the steady-state condition, the governing heat flow equations reduce to the Laplace equations shown in (1). Consequently, an $\hat{\omega}(z)$ determined for both the frozen and thawed regions satisfies the Laplace equations exactly, leaving only errors in satisfying the boundary conditions. To develop a CVBEM steady-state solution, an $\hat{\omega}(z)$ is developed for each of the separate regions. Initially, both $\hat{\omega}_f(z)$ and $\hat{\omega}_t(z)$ are defined by

$$\hat{\omega}_f(z) = \hat{\omega}_f^1, \quad z\epsilon\Omega$$
$$\hat{\omega}_t(z) = \hat{\omega}_t^1, \quad z\epsilon\Omega \qquad (4)$$

where in equation (4) $\Omega = \Omega_t \cup \Omega_f$ is the global domain, and the first-order CVBEM approximators are based on the entire domain. This procedure results in simply estimating the 0°C isotherm location for the homogeneous problem of Ω being entirely frozen or thawed. Let C^1 be the contour corresponding to this 0°C isotherm (e.g., located by a y-coordinate trial-and-error iteration such as bisection).

The second iteration step begins by defining Ω_f^2 and Ω_t^2 based on the mutual boundary of C^1. CVBEM approximators $\hat{\omega}_f^2$ and $\hat{\omega}_t^2$ are then defined for Ω_f^2 and Ω_t^2, respectively.

Examining the stream functions $\hat{\psi}_f^2$ and $\hat{\psi}_t^2$, estimates of the discrepancy in meeting equation (3) are evaluated. The $\hat{\omega}_f^2$ function is now used to determine the next location of the 0°C isotherm. This is accomplished by determining a new $\hat{\omega}_f^*$ with the stream function values of $\hat{\omega}_t^2$ (and modified to include the thawed or frozen conductivity) superimposed at the nodal values of C^1. Next, a new 0°C isotherm C^* is located for $\hat{\omega}_f^*$. The next estimated location for the 0°C isotherm, C^2, is located by averaging the y coordinates of the nodal points between C^1 and C^*. Figure 2 illustrates this procedure.

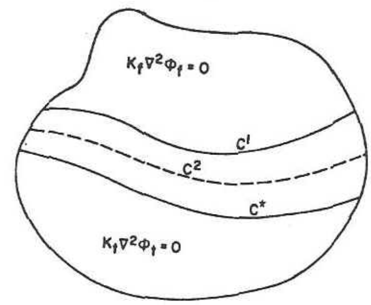

Fig. 2 Redefining the freezing front location

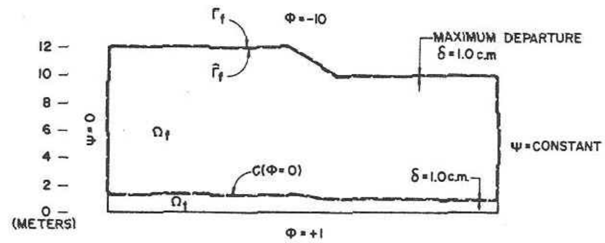

Fig. 3 The approximative boundary $\hat{\Gamma}_f$ and the goodness of fit to the problem boundary, Γ_f

The third iteration step proceeds by defining Ω_f^3 and Ω_t^3 based on the mutual boundary of C^2 and the above procedure is repeated.

The iteration process continues until the final estimates of Ω_f and Ω_t are determined with corresponding $\hat{\omega}_f$ and $\hat{\omega}_t$ approximators such that

$$|K_f d\hat{\psi}_f/ds - K_t d\hat{\psi}_t/ds| < \epsilon, \quad z\epsilon C \qquad (5)$$

The Approximative Boundary

As discussed previously, the subject problem reduces to finding a solution to the Laplace equation in Ω_f and Ω_t where Ω_f and Ω_t coincide along the steady-state freezing front location C. The CVBEM develops approximators $\hat{\omega}_f$ and $\hat{\omega}_t$ which exactly satisfy the Laplace equation over Ω_f and Ω_t, respectively. Consequently, the only numerical error occurs in matching the boundary conditions continuously on Γ_f, Γ_t, and C.

To evaluate the precision in predicting the freezing front location, an approximative boundary is determined for each subproblem domain of Ω_f, Ω_t. The approximative boundary results from plotting the level curves of each CVBEM approximator (i.e., $\hat{\omega}_f$, $\hat{\omega}_t$) which correspond to the boundary conditions of the problem.

For example, in Ω_f the thermal boundary conditions for a roadway embankment (Fig. 3) are defined on the problem boundary Γ_f by

$$\phi = -10°\text{C}, z \epsilon \text{ top surface}$$
$$\phi = 0°\text{C}, z \epsilon \text{ freezing front}$$
$$\psi = 0, z \epsilon \text{ left side (symmetry)}$$
$$\psi = \text{const}, z \epsilon \text{ right side (zero flux)}$$

After developing an $\hat{\omega}_f$ for Ω_f from the CVBEM, the approximative boundary $\hat{\Gamma}_f$ is determined by plotting the prescribed level curves. Figure 3 also includes Γ_f superimposed with Γ_f. Because $\hat{\omega}_f$ is analytic within the interior of the approximative boundary and satisfies the prescribed boundary conditions on the boundary $\hat{\Gamma}_f$, then $\hat{\omega}_f$ is the exact solution of the boundary value problem redefined on $\hat{\Gamma}_f$ and its interior $\hat{\Omega}_f$. Should $\hat{\Gamma}_f$ completely cover Γ_f, then $\hat{\omega}_f$ is the exact solution to the subject problem.

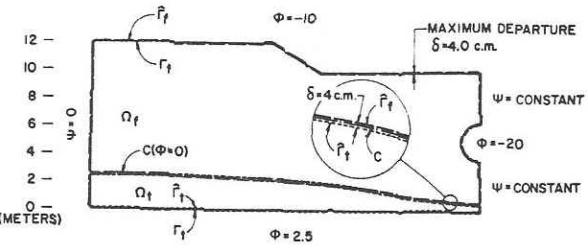

Fig. 4 Application of the CVBEM geothermal model to predict steady-state conditions

Thus, the CVBEM modeling error is directly evaluated by the closeness-of-fit between $\hat{\Gamma}_f$ and Γ_f. Hromadka and Guymon [7] successfully use this approximate boundary concept in the evaluation of approximation error resulting from the application of the CVBEM to the solution of several homogeneous potential boundary value problems. However, in this paper, the approximative boundary concept is used not only to examine the closeness-of-fit to the boundary conditions, but possibly more crucial, the closeness-of-fit of matching the estimated freezing front location between Ω_f and Ω_t along the contour C. Should Ω_f and Ω_t match C continuously, then $\hat{\omega}_f$ and $\hat{\omega}_t$ equate thermal flux continuously along C.

Applications

In order to demonstrate the use of the CVBEM approach and the approximative boundary, two applications are considered.

Figure 3 depicts an application of the geothermal model for a roadway embankment problem and the use of the approximative boundary. Figure 4 illustrates the two-dimensional steady-state freezing front location for a geothermal problem involving a buried subfreezing 3-m-dia pipeline. An examination of the approximation boundaries indicates that a good CVBEM approximator was determined by use of a 26-node CVBEM model. The maximum departure δ between the approximative boundaries and the problem boundary Γ occurred along the top of the pipeline and had a value of approximately 3.5 cm. The average departure $\bar{\delta}$ is estimated at less than 1 cm. The freezing front maximum departure is approximately 4 cm and occurred at the right-hand side. Average departure on C is less than 2 cm. (In the examples, thermal properties of a Fairbanks silt are used.)

The example problems presented illustrate the usefulness of the CVBEM in predicting the steady-state freezing front location for two-dimensional problems. Possibly the most important result is the accurate determination of the approximation error involved in using the CVBEM. The usual procedure for estimating the freezing front is to use a finite element or finite difference numerical analog. A hybrid of these domain methods is to include a variable mesh in order to better accommodate the interface. However, none of these methods provide the error of approximation. In comparison, the CVBEM model provides the approximation error not only in matching the boundary conditions, but in predicting the interface location between Ω_f and Ω_t. This error is simple to interpret as an approximative boundary displacement from the true problem boundary, and the displacement between Ω_f and Ω_t along the freezing front contour C.

Another significant result of this research is the development of the CVBEM computer code to run on many FORTRAN 64K microcomputers.

Conclusions

The CVBEM is used to develop a two-dimensional model for predicting the steady-state freezing front location. The model provides for an accurate and simple-to-use error analysis of the predicted results by use of the approximative boundary technique. The model is adaptable for many currently available microcomputers which accommodate FORTRAN.

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The Plating Algorithm for Radiation Script-F Transfer Factor

D. K. Edwards[1]

Introduction

Calculation of the script-F transfer factor is the goal of a conventional engineering radiation heat transfer analysis of an enclosure. As described in the literature, e.g., [1-7], the procedure for doing so is to use a standard matrix inversion routine followed by some matrix multiplications. An alternative view of the procedure is offered here. Here the following question is answered: How are the script-F transfer factors for an enclosure changed by "plating" a black surface, that is, changing its emissivity from 1 to its actual value? The answer is a set of recursion relations that allows the transfer factors to be found from the shape factors in N plating steps. The merits of advancing this view are thought to be as follows: (1) The recursion equations clearly display important properties that permit comparisons to be made to assess roundoff errors introduced during computation and to give added insight into the radiative transfer process. (2) The algorithm is easily coded if a personal computer at hand has no matrix inversion in its library. (3) The algorithm saves computer time in executing parametric studies.

Analysis

Transfer factors denoted here as $\bar{F}_{i,j}$ are defined conventionally so that the net flux out of an area A_i on the wall of an enclosure of N surfaces is

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