The complex variable boundary element method: Applications in determining approximative boundaries

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The complex variable boundary element method (CVBEM) is used to determine approximation functions for boundary value problems of the Laplace equation such as occurs in potential theory. By determining an approximative boundary upon which the CVBEM approximator matches the desired constant (level curves) boundary conditions, the CVBEM is found to provide the exact solution throughout the interior of the transformed problem domain. Thus, the acceptability of the CVBEM approximation is determined by the closeness-of-fit of the approximative boundary to the study problem boundary.

INTRODUCTION

Use of complex variable analytic function theory for developing approximations for two-dimensional potential problems has proved to be an effective tool for numerical analysis. Hromadka and Guymon[1] use a Cauchy integral formulation to study boundary value problems of the Laplace equation as applied to heat transfer and slow-moving boundary problems. In another paper, Hromadka and Guymon[2] generalized the boundary integral approach into a complex variable boundary element method (CVBEM).

Because the CVBEM develops an approximation function which is analytic throughout the interior of the problem domain, then necessarily the approximator solves the Laplace equation exactly throughout the interior of the domain. Consequently, the goal of the CVBEM approximation is to match the boundary condition values continuously.

Due to the imprecise trial function assumptions, the CVBEM results in an error of approximation in matching the boundary conditions. This error is manifested as a relative error distribution on the boundary. Should the boundary conditions be a set of constant values (stream or potential functions), then the CVBEM approximator can be used to locate those co-ordinates which coincide with the boundary condition values. These co-ordinates result in an approximative boundary upon which the CVBEM approximator satisfies the boundary condition values continuously, and yet satisfies the Laplace equation exactly throughout the interior enclosed by the approximative boundary. Should the approximative boundary coincide with the problem boundary, then the CVBEM approximator is the exact solution to the boundary value problem.

In this paper, the CVBEM will be briefly developed (the reader is referred to Hromadka and Guymon[2] for a complete derivation). The approximation function is defined as a Cauchy integral, resulting in a finite series of products of complex polynomials and logarithms. The CVBEM will then be applied to potential problems where the exact solutions are unknown. In order to evaluate the CVBEM approximation accuracy, an approximative boundary is determined (a discussion of the approximative boundary and its existence is given in Hromadka[3]) by matching the known boundary condition values (level curves) with the corresponding level curves of the approximator. The CVBEM approximator is the exact solution to the subject potential problem with the problem boundary transformed into the approximative boundary. The engineer can then easily evaluate the accuracy of the CVBEM approximator by visually examining the closeness of fit between the problem boundary and the approximative boundary. In this paper, the approximative boundary corresponding to a CVBEM approximation function is determined for several potential problems of interest including ideal fluid flow, groundwater seepage, and heat transfer. From the several applications it is shown that the approximative boundary concept is an effective and easy-to-use means of numerical error evaluation for the CVBEM.

CVBEM DEVELOPMENT

The CVBEM is derived in detail in Hromadka and Guymon;[2] therefore, only the major steps in developing a numerical model will be presented. Let Γ be a simple closed polygonal contour composed of straight line segments. Let Ω be the simply connected interior of Γ. Subdivide Γ into m complex variable elements (CVBE) by $\Gamma = \cup \Gamma_j$, $j = 1, 2, \ldots, m$. On each Γ_j define $(k+1)$ equidistant nodal points such that $z_{j,1}$ and $z_{j,k+1}$ are the end-points of Γ_j. Figure 1 shows the global and local nodal number conventions. The global nodal co-ordinates are related to local nodal co-ordinates by $z_{j,1} = z_j$ and $z_{j,k+1} = z_{j+1,1} = z_{j+1}$. Define complex numbers $\bar{\omega}_{ji}$ at each node z_{ji}. Then order k complex polynomials $P_j^k(z)$ are uniquely defined on each Γ_j, and an order k global trial function is defined by

$$G_k(z) = \sum_{j=1}^{m} \delta_j P_j^k(z), z \in \Gamma \qquad (1)$$

where $\delta_j = 1$ for $z \in \Gamma_j$, and $\delta_j = 0$ for $z \notin \Gamma_j$. Then $G_k(z)$ is continuous on Γ and

$$\lim_{\max|\Gamma_j| \to 0} G_k(z) = \omega(z) \qquad (2)$$

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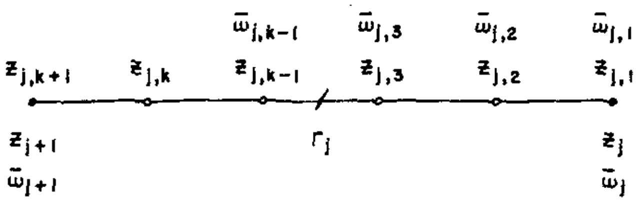

Figure 1. (k + 1)-Node boundary element Γ_j nodal definitions

where in (2) it is assumed that $\omega(z)$ is an analytic function which satisfies the potential problem on $\Omega \cup \Gamma$, and that each $\bar{\omega}_{ji} = \omega(z_{ji})$.

Consider the H_k approximation function $\hat{\omega}_k(z)$ defined by the boundary integral (in the usual positive sense of integration)

$$\hat{\omega}_k(z) = \frac{1}{2\pi i} \int_\Gamma \frac{G_k(\zeta)\, d\zeta}{\zeta - z}\,; \quad z \notin \Gamma, z \in \Omega \tag{3}$$

Using (2)

$$\int_\Gamma \frac{G_k(\zeta)\, d\zeta}{\zeta - z} = \int_\Gamma \frac{\Sigma \delta_j P_j^k(\zeta)\, d\zeta}{\zeta - z} = \sum \int_{\Gamma_j} \frac{P_j^k(\zeta)\, d\zeta}{\zeta - z} \tag{4}$$

On each Γ_j, define a local co-ordinate system by

$$\zeta_j = z_{j,1} + (z_{j,k+1} - z_{j,1})\, s_j; \quad \zeta_j \in \Gamma_j, \ 0 \leqslant s_j \leqslant 1 \tag{5}$$

Then by rearranging terms in (5) and substituting into (4)

$$\int_{\Gamma_j} \frac{P_j^k(\zeta_j)\, d\zeta_j}{\zeta_j - z} = \int_{\Gamma_j} \frac{P_j^k(s_j)\, ds_j}{s_j - \gamma_j} \tag{6}$$

where $P_j^k(s_j) = P_j^k(\zeta_j(s_j))$, and $\gamma_j = (z - z_j)/(z_{j+1} - z_j)$. Equation (6) is solved by factoring $(s_j - \gamma_j)$ from $P_j^k(s_j)$ giving

$$\int_{s_j=0}^{1} \frac{P_j^k(s_j)\, ds_j}{s_j - \gamma_j} = R_j^{k-1}(z) + P_j^k(\gamma_j) H_j \tag{7}$$

where $R_j^{k-1}(z)$ is a $k-1$ order complex polynomial; $P_j^k(\gamma_j)$ is the order k polynomial of (1) with γ_j substituted into the argument; and

$$H_j = \ln\left(\frac{d_{j+1}(z)}{d_j(z)}\right) + i\theta_{j+1,j}(z)$$

In the above, $d_j(z) = |z - z_j|$, and $\theta_{j+1,j}(z)$ is the central angle (Fig. 2) between points $\{z_j, z_{j+1}, z\}$.

Summing the m CVBE contributions along Γ gives

$$2\pi i\, \hat{\omega}_k(z) = \sum R_j^{k-1}(z) + \sum P_j^k(\gamma_j) H_j$$

Letting

$$R^{k-1}(z) = \frac{1}{2\pi i} \sum R_j^{k-1}(z),$$

the CVBEM approximation becomes

$$\hat{\omega}_k(z) = R^{k-1}(z) + \frac{1}{2\pi i} \sum P_j^k(\gamma_j) H_j \tag{8}$$

In (8), it should be noted that the $P_j^k(\gamma_j)$ are of the form of the assumed trial functions on each Γ_j.

Letting node z_1 be on the branch cut of the complex logarithm function $\ln(z - \zeta)$ where $z \in \Omega$ and $\zeta \in \Gamma$ (see Fig. 3), then (8) can be expanded as

$$\hat{\omega}_k(z) = R^{k-1}(z) - \frac{1}{2\pi i}\left\{\sum \Delta_j^{k-1}(z - z_j)\ln(z - z_j)\right\} + P_m^k(z) \tag{9}$$

where Δ_j^{k-1} is an order $(k-1)$ polynomial defined by

$$\Delta_j^{k-1} = \frac{(P_j^k(\gamma_j) - P_{j-1}^k(\gamma_{j-1}))}{(z - z_j)}$$

and $\ln(z - z_j)$ is the principal value of the complex logarithm function. From the continuity of $G^k(\zeta)$, it is seen that at nodal co-ordinate z_j, $P_j^k(\gamma_j) = P_{j-1}^k(\gamma_{j-1})$ and that $(z - z_j)$ is a factor of Δ_j^{k-1}. In (9), the $P_m^k(z)$ term appears due to

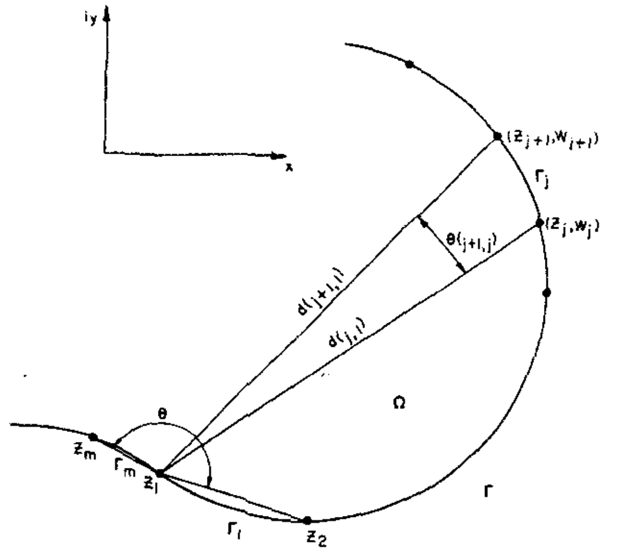

Figure 2. CVBEM linear trial function geometry for point $z = z_1$

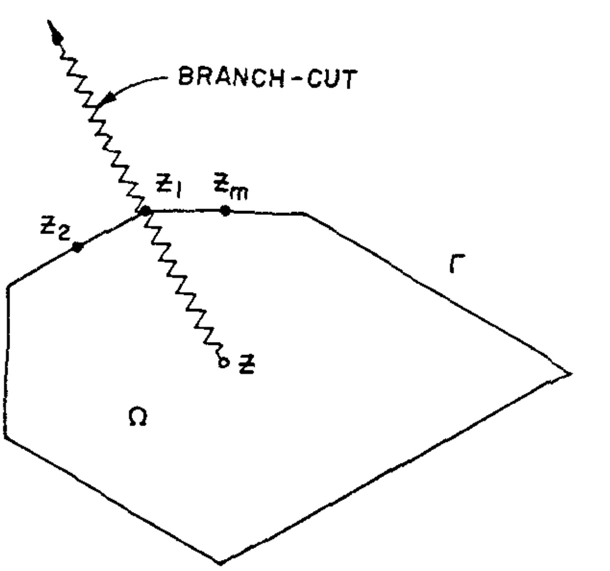

Figure 3. Branch-cut of $\ln(z - \zeta)$ function, $\zeta \in \Gamma$

the complete circuit about the branch point of the multiple valued function, $\ln(z-\zeta)$.

Letting $R^k(z) = R^{k-1}(z) + P_m^k(z)$, then the final form of the CVBEM approximator is

$$\hat{\omega}_k(z) = R^k(z) - \frac{1}{2\pi i}\sum_{j=1}^{m} \Delta_j^{k-1}(z-z_j)\ln(z-z_j) \quad (10)$$

From the form of (10), $\hat{\omega}_k(z)$ is analytic on Ω. Thus $\hat{\omega}_k(z) = \hat{\phi}(z) + i\hat{\psi}(z)$ where $\hat{\phi}(z)$ and $\hat{\psi}(z)$ are two-dimensional potential and stream functions which both solve the Laplace equation exactly on Ω. Thus, by forcing $\hat{\omega}_k(z)$ to be arbitrarily close (ϵ) to the boundary condition values of $\omega(z)$ on Γ, then it is guaranteed by the maximum modulus theorem that $|\omega(z) - \hat{\omega}(z)| \leqslant \epsilon$ for all $z \in \Omega$.

THE CVBEM APPROXIMATION FUNCTION

Because the CVBEM produces an exact solution to the Laplace equation on Ω, then convergence of $\hat{\omega}_k(z)$ to $\omega(z)$ is achieved on $\Omega \cup \Gamma$ by minimizing the approximation error on Γ. That is,

$$\lim_{\max|\Gamma_j|\to 0} \int_\Gamma \frac{G_k(\zeta)\,d\zeta}{\zeta - z} = \int_\Gamma \frac{\lim\limits_{\max|\Gamma_j|\to 0} G_k(\zeta)\,d\zeta}{\zeta - z}$$

$$= \int_\Gamma \frac{\omega(\zeta)\,d\zeta}{\zeta - z} = 2\pi i\,\omega(z) \quad (11)$$

However, we are limited to a finite number of nodal points and boundary elements on Γ. Therefore if $\omega(z)$ is not an order k (or less) complex polynomial, then

$$E(z) = R^k(z) - \frac{1}{2\pi i}\sum_{j=1}^{m} \Delta_j^{k-1}(z-z_j)\ln(z-z_j) \quad (12)$$

and a residual error $E(z)$ is to be minimized on Γ. One method of reducing $E(z)$ on Γ is to compare the known boundary condition values of $\omega(z_j)$ to the approximation values $\hat{\omega}(z_j)$ and locate regions of large deviation.[3] Additional nodal points are then specified at these large relative error locations, resulting in a reduction of $E(z)$ by adaptive integration. Specifically, a typical application of the CVBEM is to assume that at each nodal point either $\bar{\phi}$ or $\bar{\psi}$ is specified (or normal gradients of $\bar{\phi}$) and, consequently, part of the problem is to determine these unknown nodal values. For $\omega(z)$ analytic on $\Omega \cup \Gamma$ and $\bar{\omega}_{ji} = \omega(z_{ji})$, then $\bar{\phi}_{ji} + i\bar{\psi}_{ji} = \phi_{ji} + i\psi_{ji}$. The usual case is that only one value of ϕ_{ji} or ψ_{ji} is specified, thus we can write

$$\bar{\omega}_{ji} = \Delta\bar{\epsilon}_k + \Delta\bar{\epsilon}_u \quad (13)$$

where the symbol Δ is notation that $\Delta = 1$ if the associated variable is ϕ, and $\Delta = i$ if the associated varible is ψ; and k, u are notation for the known and unknown nodal values, respectively. Then the modeling strategy is to reduce the known values of $|\Delta\bar{\epsilon}_k - \Delta\hat{\epsilon}_k|$ on Γ; that is, to determine the CVBEM approximator $\hat{\omega}(z)$ such that the boundary values are arbitrarily close to the known boundary conditions, $\Delta\bar{\epsilon}_k$.

THE APPROXIMATIVE BOUNDARY

Consider the boundary value problem of the form $\nabla^2\phi = 0$ on $\Omega \cup \Gamma$ with boundary conditions of constant ϕ or ψ (or normal gradients of ϕ) along portions of Γ. Then Γ can be envisioned as the set of points z defined along the various level curves (lines of constant ϕ or ψ) composing the prescribed boundary conditions. For this type of problem, a transformation $T(z)$ exists which maps $\Gamma \cup \Omega$ onto the upper half plane, where an exact solution to the transformed problem exists. Except for a few types of boundaries, however, $T(z)$ is not readily determined.

Analogous to the above transformation, $T(z)$, the CVBEM approximator $\hat{\omega}(z)$ can be used to plot level curves corresponding to the values of the prescribed boundary conditions. The resulting approximative boundary $\hat{\Gamma}$ and its associated interior $\hat{\Omega}$ form a domain such that $\hat{\omega}(z)$ satisfies the new boundary value problem exactly on $\hat{\Omega} \cup \hat{\Gamma}$. That is, had the problem boundary been of the shape $\hat{\Gamma}$ then $\hat{\omega}(z)$ would be the exact solution. Additionally, as the distance $\|\hat{z} - z\|$ approaches zero (for $\hat{z} \in \hat{\Gamma}$ and $z \in \Gamma$), then $|\omega(z) - \hat{\omega}(z)|$ becomes negligible.[3]

Thus a more visual representation of approximation error is available by generating $\hat{\Gamma}$ and examining the closeness-of-fit to the problem boundary Γ. This approximative boundary concept is illustrated by the following potential problem applications.

Application 1. Groundwater seepage

The equation of flow continuity in a two-dimensional saturated groundwater flow regime is

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad (14)$$

where (v_x, v_y) are x and y direction soil-water flow rates, respectively. Assuming Darcy's law applies,

$$v_x = -k_x\frac{\partial\phi}{\partial x}, \quad v_y = -k_y\frac{\partial\phi}{\partial y} \quad (15)$$

where ϕ is the total energy head; and (k_x, k_y) are hydraulic conductivities in the co-ordinate axis direction. By rescaling the domain, (14) and (15) can be combined as

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \quad (16)$$

which indicates that the Laplace equation applies throughout the groundwater flow regime. Additionally, there exists a stream function ψ which is the harmonic conjugate of the potential ϕ such that the Cauchy–Riemann equations are valid

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad -\frac{\partial\phi}{\partial y} = \frac{\partial\psi}{\partial x} \quad (17)$$

Thus, there exists an analytic function $\omega(z) = \phi(z) + i\psi(z)$ which describes the groundwater flow regime.

In this application the flow regime is approximated for the case of an excavation protected by impermeable sheet-pile walls such as shown in Fig. 4. Intuitively, the sheet-pile walls necessarily define flowlines whereas the static saturated groundwater phreatic surface defines equipoentials. The base of the domain is prescribed to be impermeable (zero flux). The objective is to determine an approximator $\hat{\omega}(z)$ which equals the known boundary condition values continuously. Figure 4 shows a CVBEM (64-node) model which develops a $\hat{\omega}(z)$ approximation function. Because $\hat{\omega}(z)$ does not match the known boundary condition values continuously on Γ, an approximative boundary $\hat{\Gamma}$ (dashed lines) is determined by plotting level curves corresponding

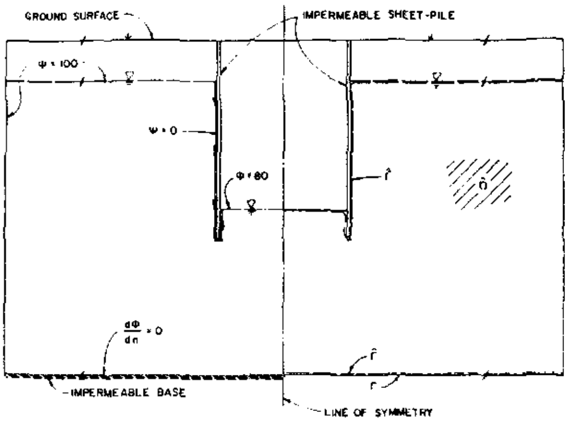

Figure 4. Application No. 1 problem definition and CVBEM approximative boundary $\hat{\Gamma}$ (dashed line)

to the prescribed boundary conditions. Figure 4 contains $\hat{\Gamma}$ superimposed on Γ. From the figure, $\hat{\omega}(z)$ is the exact solution to the problem redefined on $\hat{\Gamma}$.

Examination of the differences between Γ and $\hat{\Gamma}$ reveals that the only major discrepancies occur in the slight shortening of the sheet-pile walls (about 1%), and the rounding of all right angles on Γ. Consequently, $\hat{\Gamma}$ is an adequate geometric approximation of the true boundary, Γ, and $\hat{\omega}(z)$ is an adequate approximation of the true solution, $\omega(z)$.

Application 2. Ideal fluid flow

Ideal fluid flow in two-dimensional flow regimes is mathematically described by the Laplace equation where velocity components are defined by

$$v_x = -\frac{\partial \phi}{\partial x}, \; v_y = -\frac{\partial \phi}{\partial y} \tag{18}$$

where ϕ is a velocity potential. The CVBEM can be applied directly to approximating two-dimensional steady incompressible flow (irrotational). The corresponding stream function relates the velocity components by

$$v_x = -\frac{\partial \psi}{\partial y}, \quad v_y = \frac{\partial \psi}{\partial x} \tag{19}$$

Thus, ψ also satisfies the Laplace equation.

In this application the CVBEM is used to develop an approximation function of ideal flow down an inclined surface, over a rise of unit radius, and then onto a horizontal surface. After developing the CVBEM approximator $\hat{\omega}(z)$, an approximative boundary $\hat{\Gamma}$ is developed by plotting the level curves of $\hat{\omega}(z)$ which correspond to the boundary conditions of the problem. Figure 5 illustrates the application problem domain, boundary conditions, and a CVBEM approximative boundary $\hat{\Gamma}$ for a 48-node $\hat{\omega}(z)$ approximation.

The major geometric differences between Γ and $\hat{\Gamma}$ arise in the rounding of all right angles on Γ, and the smoothing of the circular rise. However, this smoothing seems to better approximate viscous flow effects than the abstract model of ideal flow. Consequently, the $\hat{\omega}(z)$ approximator may be considered a more appropriate model of viscous fluid flow than the true solution $\omega(z)$. Should $\hat{\Gamma}$ need to be closer to Γ, addition of nodal points is all that is required.

Application 3. Steady state heat flow

Analogous to the development of equation (16), Fourier's law relates heat flow to the spatial gradient of temperature by

$$q_x = -k_T \frac{\partial \phi}{\partial x}, \quad q_y = -k_T \frac{\partial \phi}{\partial y} \tag{20}$$

where (q_x, q_y) are the appropriate directional heat flow rates; ϕ is temperature; and k_T is the thermal conductivity. For steady state conditions, continuity of energy results in the Laplace equation. Finally, a stream function ψ exists such that the complex function $\omega(z) = \phi(z) + i\psi(z)$ is analytic throughout the problem domain.

In this application, the temperature and heat flow rate distributions are approximated by a 50-node CVBEM approximator $\hat{\omega}(z)$. Figure 6 shows the problem boundary Γ and the corresponding approximative boundary, $\hat{\Gamma}$.

This application illustrates the general tendencies of the CVBEM in approximating $\omega(z)$ on $\Omega \cup \Gamma$ with the function $\hat{\omega}(z)$ defined on $\hat{\Omega} \cup \hat{\Gamma}$. From Fig. 6, a rounding of all corners on Γ is shown on $\hat{\Gamma}$. Nevertheless, the boundary $\hat{\Gamma}$ may be adequate for most analysis purposes (with $\hat{\omega}(z)$ being the exact solution of the boundary value problem redefined on $\hat{\Omega} \cup \hat{\Gamma}$.

DISCUSSION OF PROBLEM RESULTS

The applications considered in this paper demonstrate the utility of determining an approximative boundary corre-

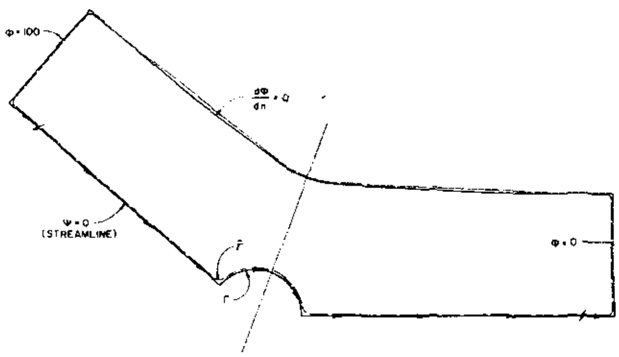

Figure 5. Ideal fluid flow problem. The approximative boundary $\hat{\Gamma}$ is plotted as dashed lines

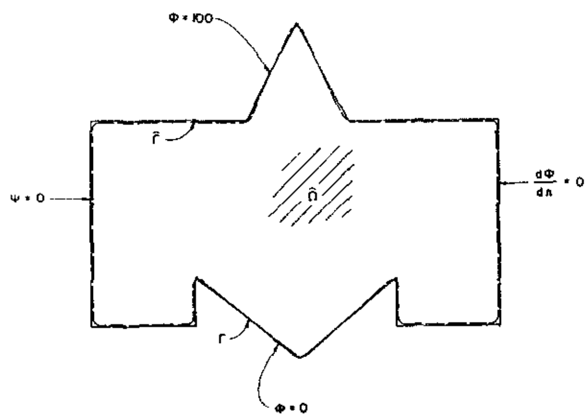

Figure 6. A steady state heat transfer problem. A CVBEM approximative boundary $\hat{\Gamma}$ is plotted as a dashed line

sponding to CVBEM approximation functions. The approximative boundary $\hat{\Gamma}$ is developed by plotting the level curves of constant potential (or stream function) which match the boundary condition values on the problem boundary Γ. Consequently, this technique is applicable only to boundary value problems which have level curves for boundary conditions.

The error of approximation is directly manifested by the departure of the approximative boundary from the problem boundary. Where large spatial discrepancies are observed, additional nodal points are added to increase the approximation accuracy. The approximative boundary can often be argued to better represent the 'as-built' or a more realistic problem boundary than the defined problem boundary. This latter idea is especially valid in large scale civil engineering studies where angle points may be actually constructed as rounded edges.

To illustrate the CVBEM approximation results within the interior of the problem domain, Figs. 7 and 8 show groundwater seepage problems with the approximated boundary, and several streamlines and lines of the constant potential plotted. Because the maximum approximation error magnitude ϵ must occur on the boundary, interior values of $\hat{\omega}(z)$ necessarily differ from $\omega(z)$ (in magnitude) by less than ϵ.

CONCLUSIONS

The CVBEM develops an approximation of the analytic solution function $\omega(z)$ defined on $\Omega \cup \Gamma$. The CVBEM approximator $\hat{\omega}(z)$, however, is the exact solution to the Laplace equation boundary value problem redefined on $\hat{\Omega} \cup \hat{\Gamma}$. Consequently, the success of the CVBEM can be readily inspected by evaluating the closeness-of-fit of the approximative boundary $\hat{\Gamma}$ to the problem boundary Γ.

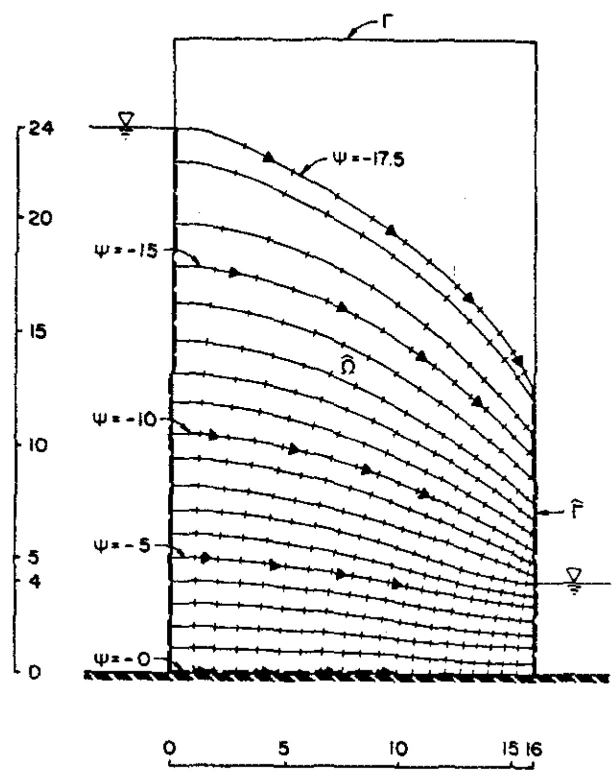

Figure 7. Plot of streamlines and potentials for soil-water flow through a homogeneous soil

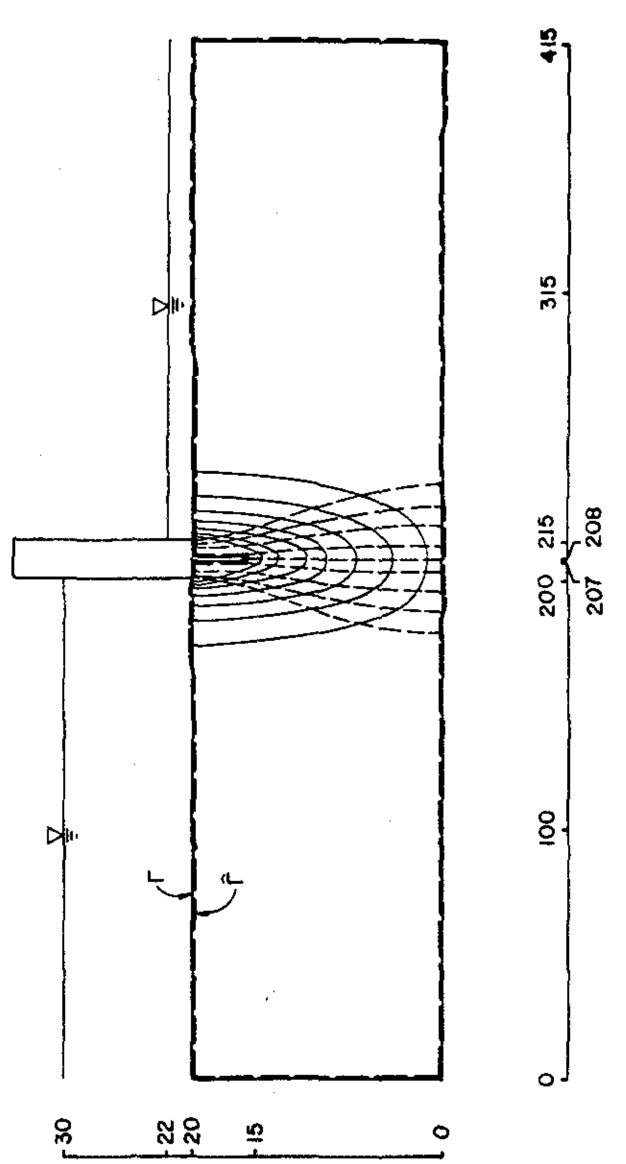

Figure 8. Plot of streamlines and potentials for soil-water flow beneath a dam. (Note that the vertical and horizontal scales differ)

ACKNOWLEDGMENTS

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