ADJUSTING THE NODAL POINT DISTRIBUTION IN DOMAIN GROUNDWATER FLOW NUMERICAL MODELS

T. V. Hromadka II and T. J. Durbin

U.S. Geological Survey

ABSTRACT

A procedure for anticipating the relative error which results from a domain model of groundwater flow is presented. The procedure uses the complex variable boundary element method to develop steady-state solutions within the groundwater basin (or portions of the basin). These steady-state solutions are used for comparison with the domain model approximation, and a relative error distribution is determined. The nodal point distribution is adjusted to increase the nodal density in regions of large error, and likewise decrease the nodal density in regions of small error. The resulting nodal distribution is found to produce associated transient problem solutions comparable to approximations generated by increasing the total number of nodal points.

INTRODUCTION

A popular method for approximating groundwater flow effects is by means of numerical modeling. Generally, domain methods such as finite element and finite difference are used, although collocation methods and boundary integral equation methods have also been employed. In the domain methods the problem domain is discretized by nodal points into control volumes or finite elements. The choice as to nodal point placement is usually based on the judgment and experience of the hydrologist. Generally, the nodal point density is increased in regions where the state variable (e.g. water surface) is anticipated to vary rapidly with respect to either space or time. Additional placement of nodal points is governed by the interface between dissimilar materials or boundary condition specifications (e.g. aquifer fractures, groundwater wells, etc.). A complete presentation of such domain numerical methods is given in Finder and Gray (1977).

In this paper, the main objective is to report on a procedure for identifying regions within the problem domain where the nodal point density needs to be increased in order to increase the numerical accuracy. The basis of the procedure is to examine the accuracy of the numerical model in predicting steady-state conditions where various boundary value problem conditions are considered. In order to examine the steady-state predicted values, the CVBEM (complex variable boundary element method) (Hromadka and Guymon, 1984a; Hromadka, 1984) is used to develop nodal point approximation values and estimates of nodal point relative error. Nodal points are then added (or removed when possible) in regions where the domain model estimates of the steady-state values differ significantly from the CVBEM predicted values. In this fashion, the conduction process modeling error due to choice of discretization is reduced.

Only the two-dimensional discretization of the groundwater basin is considered where all vertical direction effects are averaged by integration. The procedures can be extended to the three-dimensional case by examining several two-dimensional cross sections of the problem domain (such as vertical and horizontal slices of the domain).

The paper is divided into four parts. The first part presents the development for approximation of the transient groundwater flow regime. The NDI (nodal domain integration technique) (Hromadka, et al., 1981) is used to develop the domain numerical solution. An advantage of using the NDI approach is that the Galerkin finite element method, the subdomain integration method, and an integrated finite difference method can be represented by a single computer code. The second part of the paper presents a brief development of the CVBEM for use in determining a highly accurate solution of the two-dimensional Laplace equation. Because the CVBEM affords an immediate and exact evaluation of approximation error and results in an approximation function which exactly solves the Laplace equation, the numerical technique can be used to determine (to a high level of accuracy) an approximation of a Laplace equation boundary value problem. Part three presents the error evaluation technique (Hromadka and Guymon, 1984b) used with the CVBEM. Finally, part four presents two illustrations of the presented approach in locating domain method nodal points.

DOMAIN NUMERICAL MODEL DEVELOPMENT

Two-dimensional models of groundwater basins have been extensively reported in the literature. Generally, either the finite element or finite difference method is used to develop a matrix system of nodal point values as functions of the basin geometry, flow parameters, and boundary conditions. The general equation solved is to the form (for a confined aquifer)

$$\frac{\partial}{\partial x}\left[K_x \frac{\partial \phi}{\partial x}\right] + \frac{\partial}{\partial y}\left[K_y \frac{\partial \phi}{\partial y}\right] = S \frac{\partial \phi}{\partial t} \tag{1}$$

where

ϕ = total energy head with respect to a reference elevation

k_x, k_y = x- and y-direction hydraulic conductivities

S = specific storage

For homogeneous, isotropic aquifers (1) can be rewritten as

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{S}{bK} \frac{\partial \phi}{\partial t} \tag{2}$$

Application of the domain numerical method results in a matrix system

$$[K] \underset{\sim}{\phi} + [C] \dot{\underset{\sim}{\phi}} = \underset{\sim}{F} \tag{3}$$

where [k] is a symmetrical banded matrix representing the soil-water flowrates from the nodal point control volumes; [C] is a symmetrical banded matrix representing the capacitance of the nodal point control volumes; $\underset{\sim}{F}$ is a vector of specified nodal point values and flux boundary conditions (with [K] and [C] appropriately modified); and $\underset{\sim}{\phi}$ and $\dot{\underset{\sim}{\phi}}$ are the vectors of nodal point values and their time derivatives. Hromadka, et al. (1981) show that an infinity of domain methods can be described by (3) when written in the form

$$[K] \underset{\sim}{\phi} + [C(\eta)] \dot{\underset{\sim}{\phi}} = \underset{\sim}{F} \tag{4}$$

where η = 2, 22/7, ∞ results in the Galerkin analog, subdomain integration, and an integrated finite difference formulation, respectively.

In this paper, only errors in approximating the flowrates are considered. That is, it is assumed that the integration approximation scheme for the state variable in each nodal point control volume is assumed adequate (i.e., an appropriate η exists in (4)), and the specified flow parameters and boundary conditions are assumed adequately defined. To evaluate the numerical errors resulting from the [K] matrix, a steady-state problem is solved of the form

$$[K] \underset{\sim}{\phi} = \hat{\underset{\sim}{F}} \tag{5}$$

where $\hat{\underset{\sim}{F}}$ is a vector representing the boundary conditions for a selected steady-state scenario. Usually, several groundwater basin scenarios are considered resulting in several approximations from (5) which can be examined for numerical error de-

velopment. However, to evaluate the error in (5), the ϕ vector needs to be compared to the correct solution vector ϕ^*. Because an analytic solution for the problem approximated by (5) is seldom available, the CVBEM is used to develop another approximation vector ϕ' and a corresponding relative error distribution. The ϕ' values represent a highly accurate estimate of the exact solution values, ϕ. The ϕ' vector is then used for comparison purposes with the domain model solution of ϕ in order to locate regions where the domain method approximation deviates substantially from the CVBEM approximation values.

CVBEM DEVELOPMENT

Hromadka and Guymon (1984a) present a detailed development of the CVBEM. A comprehensive presentation of the method is given in Hromadka (1984). A feature available with the CVBEM is the generation of a relative error measure which can be used to match the known boundary condition values of the problem. Consequently, the method can be used to develop a highly accurate approximation function for the Laplace equation and yet provide a descriptive relative error distribution for analysis purposes. Because the main objective of this paper is to analyze the numerical error in solving (5), it is noted that the Laplace equation is solved throughout the problem domain (if homogeneous) or in connected subregions (if inhomogeneous). Many anisotropic effects can be accommodated by the usual rescaling procedures or by subdividing the total domain into easier-to-handle subproblems. The CVBEM is then applied to the problem domain(s) as discussed in the following.

Let Ω be a simply connected domain with boundary Γ where Γ is a simple closed contour (Figure 1). Discretize Γ by m nodal points into m boundary elements such that a node is placed at every angle point on Γ (Figure 2). Each boundary element is defined by

$$\Gamma_j = \{z: z = z(s) \text{ where } z(s) = z_j + (z_{j+1} - z_j)s,\ 0 \leq s \leq 1\},\ j \neq m \quad (6)$$

with the exception that on the last element,

$$\Gamma_m = \{z: z = z(s) \text{ where } z(s) = z_m + (z_1 - z_m)s,\ 0 \leq s \leq 1\}$$

Then

$$\Gamma = \bigcup_{j=1}^{m} \Gamma_j \quad (7)$$

Let each Γ_j be discretized by (k+1) evenly spaced nodes ($k \geq 1$) such that Γ_j is subdivided into k equilength segments (Figure 3). Then Γ_j is said to be a (k+1)-node element. From

Figure 3, each Γ_j has an associated nodal coordinate system such that $z_{j,1} = z_j$ and $z_{j,k+1} = z_{j+1} = z_{j+1,1}$.

On each Γ_j, define a local coordinate system by

$$\begin{aligned}\zeta_j(s) &= z_{j,1} + (z_{j,k+1} - z_{j,1})s, \quad 0 \le s \le 1 \\ &= z_j + (z_{j+1} - z_j)s\end{aligned} \tag{8}$$

where $d\zeta_j = (z_{j,k+1} - z_{j,1})ds$.

On each (k+1)-node element Γ_j, a set of order k polynomial basis functions are uniquely defined by

$$N^k_{j,i}(s) = a_{j,i,0} + a_{j,i,1}\, s + \cdots + a_{j,i,k}\, s^k \tag{9}$$

where $i = 1,2,\cdots,(k+1)$ and $0 \le s \le 1$, and where

$$N^k_{j,k}\left(\frac{z_{j,n} - z_{j,1}}{z_{j,k+1} - z_{j,1}} \right) = \begin{cases} 1, & n = i \\ 0, & n \neq i \end{cases} \tag{10}$$

The basis functions are further defined to have the property that for $\zeta \in \Gamma$

$$N^k_{j,i}\left(\frac{\zeta - z_{j,1}}{z_{j,k+1} - z_{j,1}} \right) = \begin{cases} N^k_{j,i}\left(\dfrac{\zeta - z_{j,1}}{z_{j,k+1} - z_{j,1}} \right), & \zeta \in \Gamma_j \\ 0, & \zeta \notin \Gamma_j \end{cases} \tag{11}$$

Let $\omega(z)$ be analytic on $\Omega \cup \Gamma$. That is, let $\omega(z)$ be the solution (unknown) to the steady-state boundary condition problem being used to study (5). At each nodal point on Γ, define a specified nodal value by (Figure 3)

$$\bar{\omega}_{j,i} = \omega(z_{j,i}) \tag{12}$$

where from Figure 3, $\bar{\omega}_{j,1} = \bar{\omega}_j = \bar{\omega}_{j-1,k+1}$.

Using (11) and (12), an order k global trial function is defined by

$$G^k(\zeta) = \sum_j G^k(\zeta_j(s)) = \sum_j \sum_{i=1}^{k} \bar{\omega}_{j,i} N_{j,i}^k \left(\frac{\zeta - z_j}{z_{j+1} - z_j} \right) \tag{13}$$

From (13), the global trial function is continuous on Γ. An H_k approximation function $\hat{\omega}_k(z)$ (Hromadka, 1984) is defined by the Cauchy integral

$$\hat{\omega}_k(z) = \frac{1}{2\pi i} \int_\Gamma \frac{G^k(\zeta)\, d\zeta}{\zeta - z}, \quad z \in \Omega, \quad z \notin \Gamma \tag{14}$$

Because the derivative of $\hat{\omega}_k(z)$ exists for all $z \in \Omega$, then $\hat{\omega}_k(z)$ is analytic in Ω and exactly solves the Laplace equation in Ω.

Expanding (14) and using (7) gives

$$\int_\Gamma \frac{G^k(\zeta)\, d\zeta}{\zeta - z} = \sum_{j=1}^{m} \int_{\Gamma_j} \frac{G^k(\zeta)\, d\zeta}{\zeta - z} \tag{15}$$

Integrating on boundary element j gives (Hromadka, 1984)

$$\int_{\Gamma_j} \frac{G^k(\zeta)\, d\zeta}{\zeta - z} = R_j^{k-1}(z) + \sum_{i=1}^{k} \bar{\omega}_{j,i} N_{j,i}^k(\gamma_j) \ln \left(\frac{z - z_{j+1}}{z - z_j} \right) \tag{16}$$

where $R_j^{k-1}(z)$ is an order (k-1) complex polynomial resulting from the circuit around point z (Figure 4) and γ_j is equal to $(z - z_j)/(z_{j+1} - z_j)$.

Thus, the CVBEM results in the approximation function

$$\hat{\omega}_k(z) = \frac{1}{2\pi i} \sum_j \left[R_j^{k-1}(z) + \sum_i \bar{\omega}_{j,i} N_{j,i}^k(\gamma_j) \ln \left(\frac{z - z_{j+1}}{z - z_j} \right) \right] \tag{17}$$

or in a simpler form (Hromadka, 1984)

$$\hat{\omega}_k(z) = R^k(z) + \frac{1}{2\pi i} \sum_j \ln (z - z_j) \sum_i T_i^k \tag{18}$$

where $T_i^k = \bar{\omega}_{j-1,i} N_{j-1,i}^k (\gamma_{j-1}) - \bar{\omega}_{j,i} N_{j,i}^k (\gamma_j)$, and $R^k(z)$ follows from (17).

The approximation function of (18) exactly satisfies the governing flow equation in the problem domain Ω for the approximated boundary conditions on the problem boundary, Γ. Because $\hat{\omega}_k(z)$ is analytic on Ω, then the maximum relative error of $|\omega(z) - \hat{\omega}_k(z)|$ must occur on Γ. Consequently, the total approximation error can be simply evaluated on Γ with the corresponding errors in the interior of Ω being less in magnitude. Because the boundary conditions used to evaluate (18) are known continuously on Γ, then $\hat{\omega}_k(z)$ can be determined within arbitrary accuracy by the addition of nodal points on Γ due to (without proof)

$$2\pi i \lim_{\max|\Gamma_j|\to 0} \hat{\omega}_k(z) = \int_\Gamma \frac{\lim_{\max|\Gamma_j|\to 0} G^k(\zeta)\, d\zeta}{\zeta - z} = \int_\Gamma \frac{\omega(\zeta)\, d\zeta}{\zeta - z} = 2\pi i\, \omega(z) \quad (19)$$

CVBEM ERROR REDUCTION TECHNIQUE

The previous discussion shows that the resulting CVBEM approximation function $\hat{\omega}(z)$ is analytic in the problem domain. Therefore, both the real equipotential approximation function $\hat{\phi}$ and the imaginary approximation $\hat{\psi}$ function satisfy the Laplace equation $\nabla^2\hat{\phi} = \nabla^2\hat{\psi} = 0$ for $z \in \Omega$, and $\hat{\omega} = \hat{\phi} + i\hat{\psi}$.

Therefore, the only approximation remaining is the continuous matching of the boundary conditions on the problem boundary, Γ.

This error of approximation is reflected in the difference between $\hat{\omega}(z)$ and $\omega(z)$ on Γ for the conjugate function specified. For example, if ϕ is known on element Γ_1, then the error used is $e_\phi = \phi - \hat{\phi}$ for $z \in \Gamma_1$. Likewise, if ψ is known on element Γ_2 then the error used is $e_\psi = \psi - \hat{\psi}$. This relative error can be determined continuously on Γ and an appropriate plot of the error in matching the boundary constructed. Hromadka and Guymon (1984b) use this error distribution on Γ to locate additional collocation points on Γ in order to develop a subsequent refined CVBEM approximation function. In this fashion, a highly accurate approximation of the boundary conditions is achieved on Γ while still retaining an exact solution of the governing partial differential equation (Laplace equation) in Ω.

After developing a $\hat{\omega}(z)$ which has an acceptable error distribution on Γ, the approximation function is then used as a sub-

stitute for the exact solution of the boundary value problem, $\omega(z)$.

From the maximum modulus theorem, the maximum value of $M = |\omega(z) - \hat{\omega}(z)|$ occurs for some point $z^* \in \Gamma$. Thus, the CVBEM approximation values for points in the interior of Ω (such as nodal points specified for the discretization of Ω for the NDI model) are guaranteed to have an approximation error less than M.

It is stressed that the principal step in the proposed procedure is the development of an accurate CVBEM approximation function. Generally, two to four attempts are required to develop a $\hat{\omega}(z)$ which has associated e_ϕ and e_ψ magnitudes which can be considered negligible.

APPLICATION

After developing the CVBEM $\hat{\omega}(z)$ function, the accuracy of the NDI domain model can be tested in the solution of the steady-state problems. Using identical boundary conditions on Γ, the CVBEM and the NDI models are compared as to the discrepancy in nodal values for all nodes located in the interior of Ω. The variation in modeling estimates determines a distribution of the approximation error throughout Ω. At regions of large error in Ω, additional nodal points are added to the domain discretization. Continuing in this fashion, the approximation error due to the nodal point distribution in Ω is reduced.

In the following application problems, the domain model is initially applied to a uniform distribution of nodal points in Ω. That is, the density of nodal points ρ_N (number of nodes per unit area) is a constant. After developing a $\hat{\omega}(z)$ function, the error distribution E(z) between the CVBEM and NDI models is determined in Ω. The next step is to redistribute the NDI interior nodes such that the nodal density ρ_N variation is related to an error density distribution ρ_E which is defined by

$$\rho_E(z_j) = \int_{\Omega_j} |E| dA \Big/ \int_{\Omega} |E| dA \qquad (20)$$

where z_j is a nodal point coordinate; Ω_j is the domain control volume associated to nodal point z_j; and $|E|$ is the magnitude of the error between the CVBEM and NDI models. The $\rho_E(z_j)$ values may be estimated as simple point values. In the applications, $\rho_N(z_j)$ is reset by letting

$$\rho_N = \rho_E \text{ (total number of nodes)}. \qquad (21)$$

It is noted that equations (20) and (21) are arbitrary and other relations could be used. It is also noted that in the applications presented, the total number of nodes is assumed fixed, and only a redistribution of nodes is performed. Generally, the analyst would simply add nodes to regions in Ω where the error distribution is large.

Application I

A confined groundwater aquifer of thickness 30.5 m and with flow parameters K = 30.5 m/day and S = 0.01 has a centrally located well. The problem domain is a 360 m radius field with the well located at the centroid (Figure 5). The problem is to calculate the drawdown within the confined aquifer due to a constant pumping of Q = 1.7 m^3/min at the well.

In order to numerically model the problem, the domain has to be discretized by nodal points. Figure 5 shows a uniform nodal point distribution. Assuming a typical drawdown at the well and a uniform constant head boundary condition along the problem domain, the CVBEM is now used to estimate the drawdown values throughout the domain. Comparing the CVBEM values to estimates of drawdown (steady-state conditions) obtained from the domain model of (5) indicates a nearly uniform increase in relative error as the distance to the well decreases (Figure 6).

The problem domain is rediscretized by the nodal densities shown in Figure 7. Figure 8 shows the estimated relative error for steady-state conditions using the selected nodal point densities of Figure 7.

In order to check the performance of the two discretizations in the transient problem, the drawdown can be checked along a radial (for a short simulation time) by using the well-known Theis solution. Figure 9 shows the plots of drawdown obtained by both discretizations along with the drawdown predicted by the Theis analytic solution. (In the domain model, the Crank-Nicolson time advancement algorithm is used with a timestep of 0.1 days.) From the Theis solution, the adjusted nodal point distribution provides a significant reduction in error.

In this application problem, the nodal point density is determined to be increased near the vicinity of the well. The motivation for adjusting the nodal point distribution is to reduce the relative error obtained in solving a steady-state problem which approximates the range of conditions that are anticipated for the associated transient problem. The key to this approach is determining the steady-state problem approximation error in order to evaluate the nodal point densities. Because the CVBEM provides a highly accurate approximation for the Laplace equation, it can be used for determining the

relative error by comparison with the results generated by the domain model.

Application II

The above procedure can be applied to improving the nodal point distribution in steady-state problems which are governed by the Laplace equation. The domain problem shown in Figure 10 is initially discretized using the shown nodal point distribution. Of concern is the pressure distribution along the specified pressure surface. Figure 11 shows the relative error (percent) in comparison to the CVBEM predicted values. Using the error distribution as a guide, the 40 nodal points were redistributed to be concentrated more in regions of large relative error. Resolving the domain method steady-state problem produced a significant decrease in relative error along the pressure surface. In comparison, increasing the original 40-node distribution uniformly to 120 nodes resulted in a relative error along the pressure surface of comparable magnitude to the redistributed 40-node model.

The adjusted 40-node distribution and the 120-node models were used to examine several boundary conditions of various water depths upstream and downstream of the dam. In all test trials, both models produced comparable results which indicates that the adjusted nodal point distribution is not significantly affected by changes in magnitude of the problem boundary conditions.

CONCLUSIONS

In using domain models for estimating groundwater flow effects, the choice as to the nodal point distribution is usually left to the experience of the hydrologist. Typically, nodal densities are increased in regions where it is anticipated that there may be a large departure from the assumed trial function distribution within a nodal point's associated control volume.

In this paper, a procedure to examine the nodal point distribution is proposed. Using steady-state conditions in portions of the problem domain, the CVBEM is used to generate a solution of the Laplace equation for comparison with the analogous domain model approximation. Based on the accompanying relative error, the domain model nodal distribution is adjusted to have an increased density in regions of large error between the models.

The adjusted nodal distribution is found to significantly reduce modeling error for the subsequent transient problems, and is not significantly affected by changes in the boundary conditions.

The two applications problems illustrate the proposed procedure

in studying the nodal point distributions for two classic problems. For large-scale problems where available computer memory becomes an issue, the procedure can be used to aid in optimizing the nodal point distribution throughout the problem domain. In this way, nodal point densities can be increased in regions of large anticipated relative error and, perhaps more important, decreased in regions of small relative error with the end result of keeping the computer memory requirements at a minimum.

REFERENCES

Hromadka II, T., Guymon, G., and Pardoen, G. (1981) Nodal Domain Integration Model of Unsaturated Two-Dimensional Soil-Water Flow: Development. Water Resources Research, (17), 1425-1430.

Hromadka II, T. (1984) The Complex Variable Boundary Element Method. Springer-Verlag publishers (in press).

Hromadka II, T., and Guymon, G. (1984a) The Complex Variable Boundary Element Method: Development. Int. Jour. Num. Methods in Eng. (in press).

Hromadka II, T., and Guymon, G. (1984b) An Alogrithm to Reduce Approximation Error from the CVBEM Applied to Soil Freezing. Numerical Heat Transfer (in press).

Pinder, G., and Gray, W. (1977) Finite Element Simulation in Surface and Subsurface Hydrology. Academic Press.

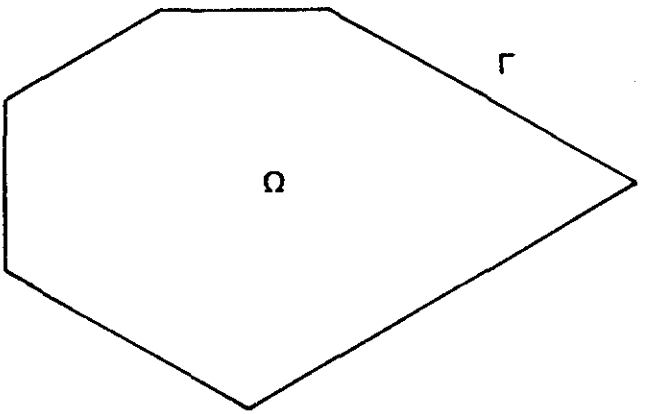


Figure 1. Simply Connected Domain Ω with Simple Closed Contour Boundary Γ

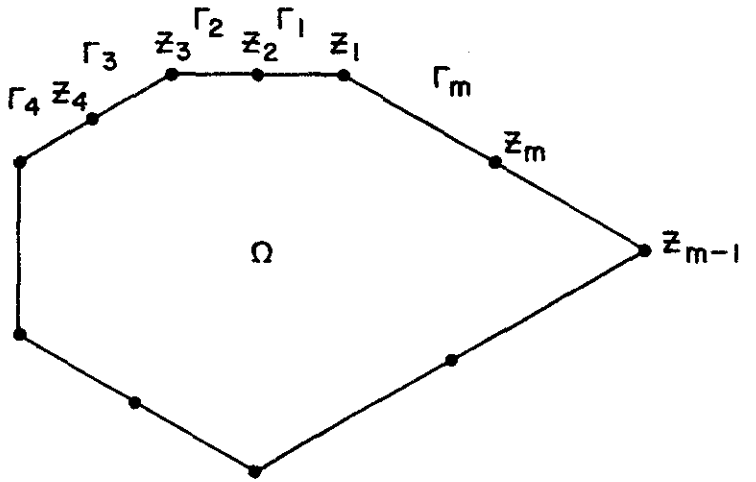
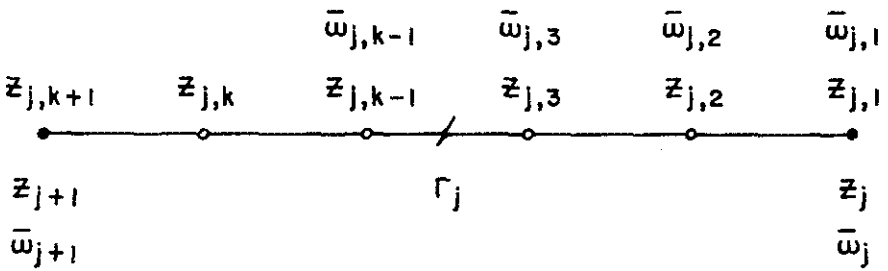


Figure 2. Γ Discretized Into m Boundary Elements



LEGEND

- Element Endnode
- Element Interior Node

Figure 3. (k+1)-Node Boundary Element Γ_j Nodal Definitions

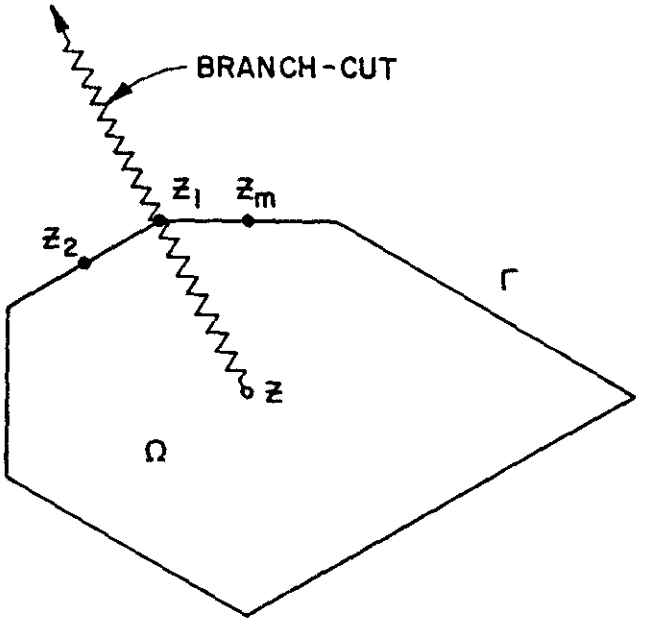


Figure 4. Branch-Cut of $\ln(z - \zeta)$ function, $\zeta \in \Gamma$

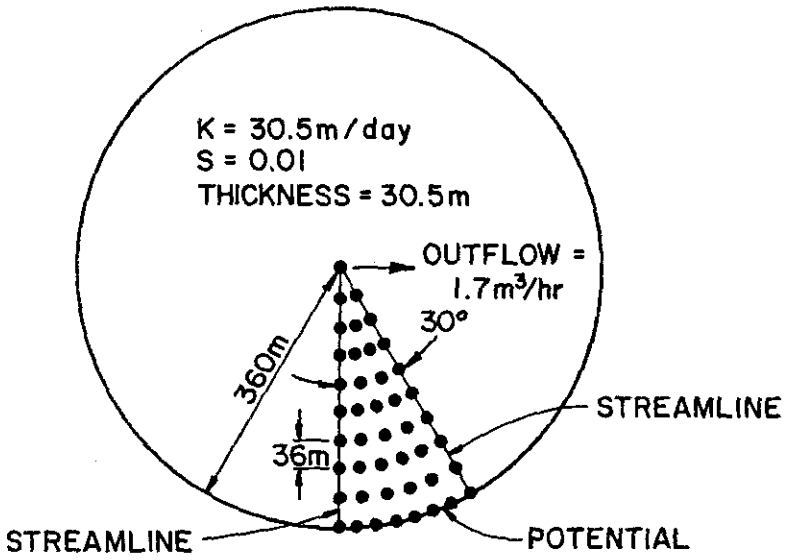


Figure 5. Problem Domain Definition and Uniform Nodal Point Distribution

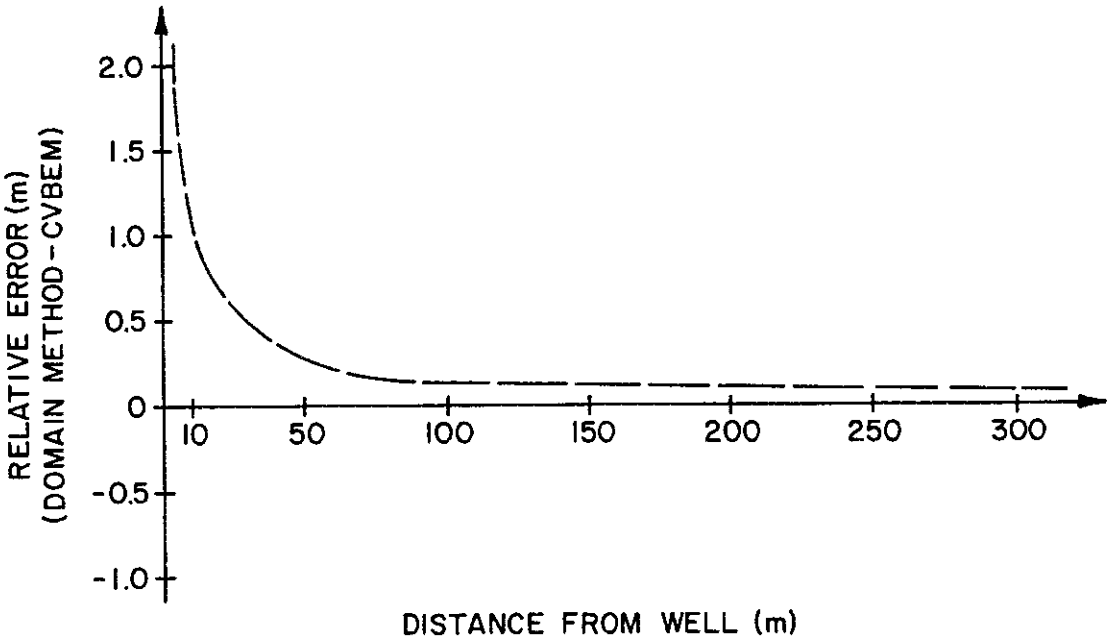


Figure 6. Relative Error Using Uniform Radial Nodal Point Density

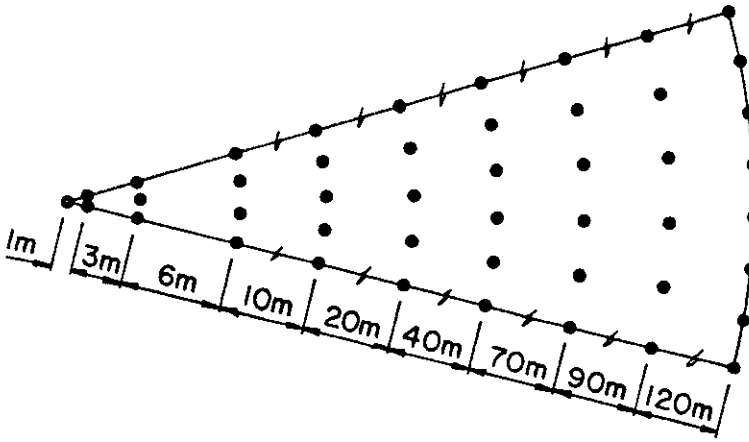


Figure 7. Adjusted Nodal Point Distribution

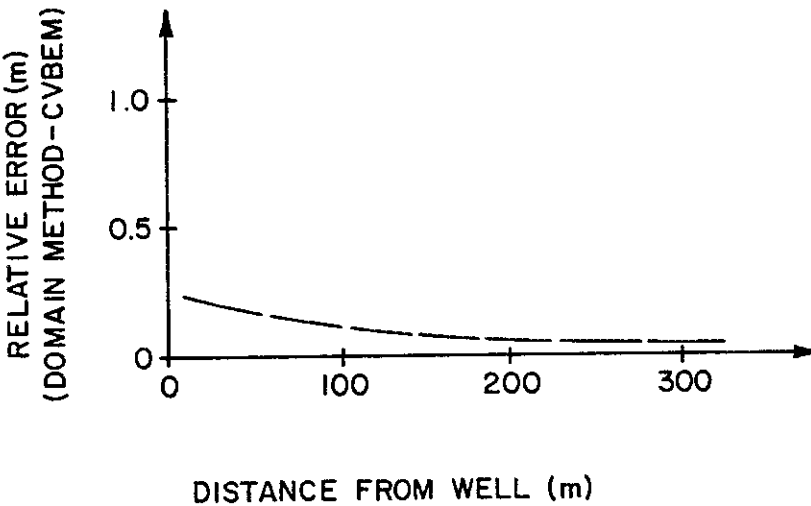


Figure 8. Relative Error Using Adjusted Nodal Point Distribution

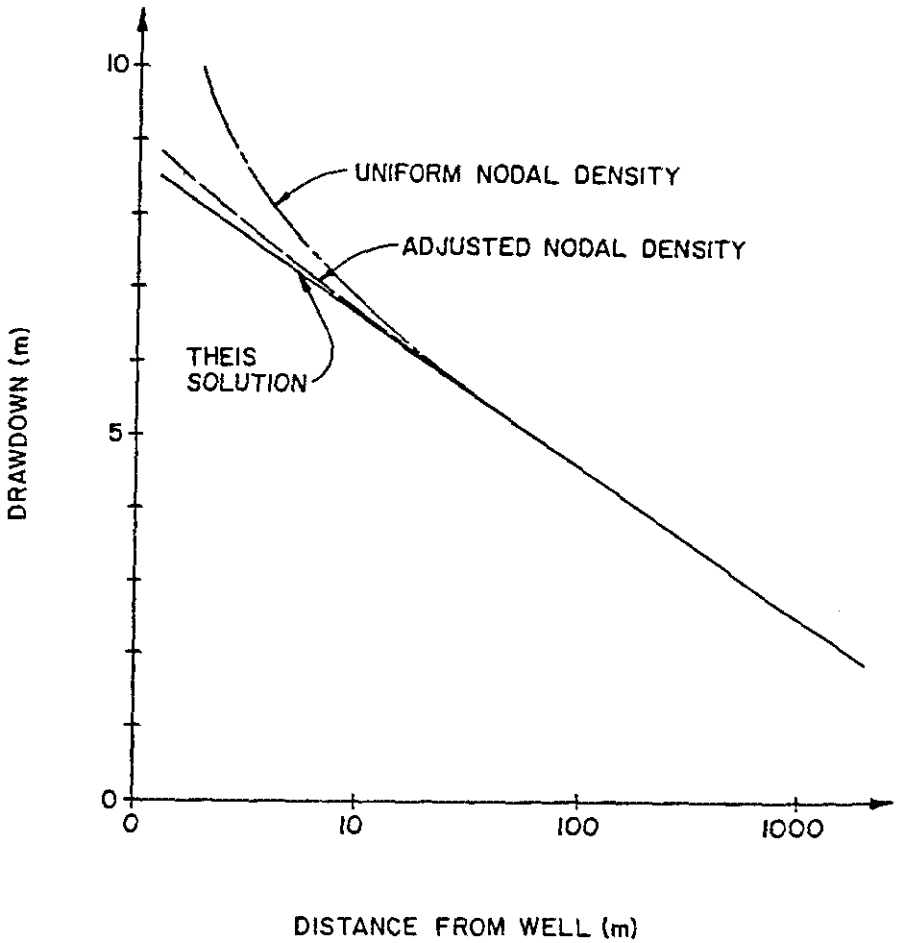


Fig. 9. Radial Drawdown at time = 0.5 days

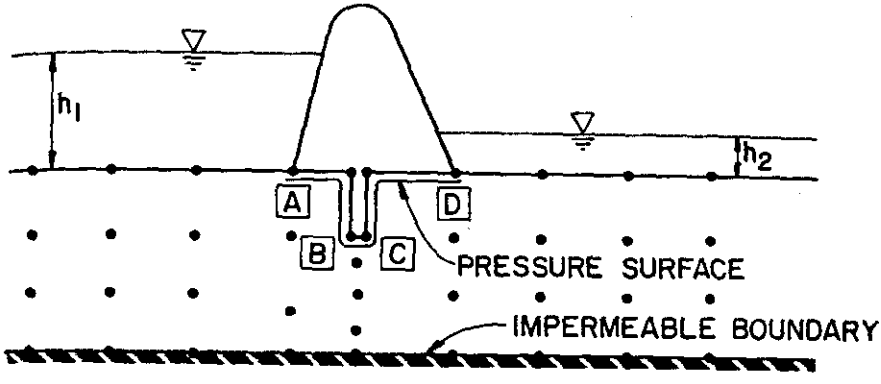


Figure 10. Flowfield Approximated Geometry

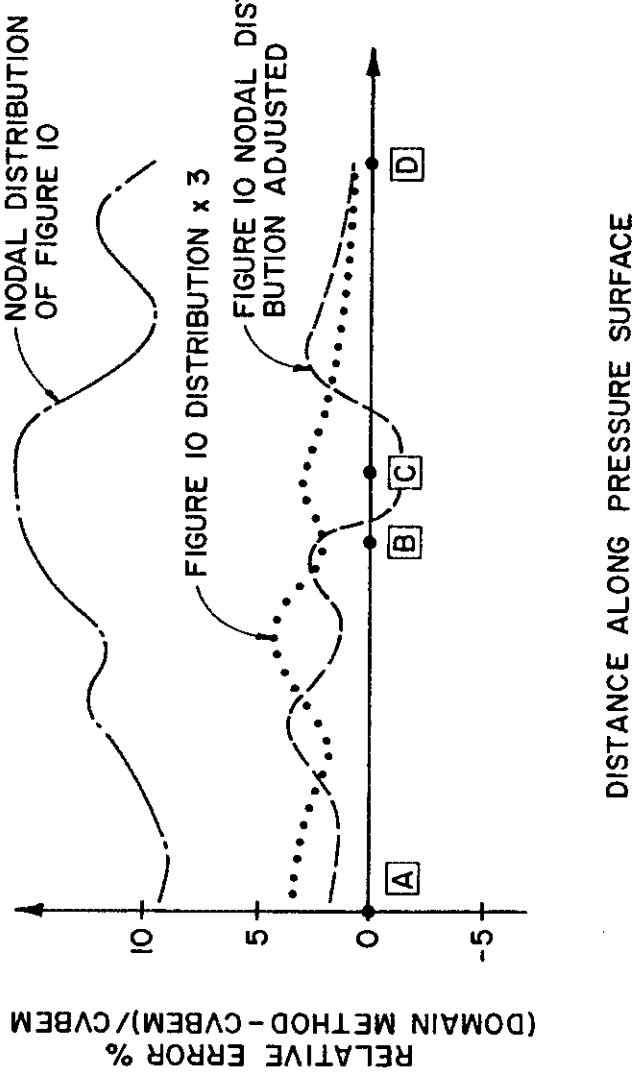


Figure 11. Relative Error Plots for the Flowfield Problem of Figure 10.