A simple model of a phreatic surface through an earth dam

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A simple numerical model for estimating a phreatic surface in an earthen dam is presented. The numerical approach is based upon the Complex Variable Boundary Element Method (CVBEM). By expanding the CVBEM approximation geometric functions into a first order Taylor series, the unknown phreatic surface location geometrics can be approximated without iteration by solving a single matrix system. The developed technique provides for the numerical solution of the inverse problem of locating the phreatic surface coordinates. A comparison of results produced from this simple approach to results produced from a finite element analog and an iterative CVBEM analog for an example problem is presented.

INTRODUCTION

The classic problem of estimating the free water (phreatic) surface in a homogeneous isotropic soil has been the subject of several papers and reports in the literature. In all cases, either a domain method (such as finite difference or finite element) or a boundary element technique (such as a boundary integration equation formulation or complex variable approach) is used to develop the approximate seepage face and corresponding approximate water-surface elevations, and an iteration procedure is used which adjusts the phreatic surface elevations until the predicted potentials coincide with the surface elevations.

In this note, a simple procedure for estimating the phreatic surface without an iterative procedure is presented. The approach is also based on the Complex Variable Boundary Element Method, or CVBEM,[1] but produces estimates for the phreatic surface elevations with a single matrix solution, as against by iteration in the previous approaches.

MODEL DEVELOPMENT

The CVBEM procedure is developed in detail in Hromadka and Guymon.[1] The numerical method develops an approximation function which is analytic in its simple connected complex domain of definition which is enclosed by a simple closed contour boundary. The basis of the method is the Cauchy integral theorem which determines the values of an interior point from a line integral of the values along the problem boundary. Because the approximator is analytic, the numerical model solves the Laplacian exactly for various boundary condition configurations. Because the seepage problem can be mathematically modeled by the Laplace equation, it is possible to apply the CVBEM directly to the subject problem.

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PROBLEM DEFINITION

In this paper, the classic problem of groundwater flow (with a free surface) through a homogeneous earthen embankment is modeled using the CVBEM. The main objective is to determine the phreatic surface by solving directly for the free surface coordinates rather than iterating between the elevation, y, and the flow potential, ϕ, until $\phi = y$. A boundary condition for the CVBEM model is the phreatic surface estimated from using the Dupuit approximation.[2] The problem considered[3] is described as follows (Fig. 1):

$$\nabla^2\phi = \nabla^2\psi = 0, \quad (x,y)\in\Omega \tag{1}$$

with boundary conditions:

$$\psi = -17.5;\ \phi = y, (x,y)\in\Gamma_S$$

$$\psi \text{ is linear};\ \phi = 24, (x = 0, y) \in \Gamma_L$$

$$\psi \text{ is linear};\ \phi = 4, (x = 16, y)\in\Gamma_R$$

$$\psi = 0, (x,y)\in\Gamma_b$$

where ϕ is the total energy head and ψ is the stream function. On Γ_S, the phreatic surface, $\psi = -17.5$ is determined from the Dupuit-Forchheimer model[4] of the seepage problem free water surface $h(x)$ approximated by the parabola:

$$h^2(x) = -35x + 576, \quad 0 \leqslant x \leqslant 16 \tag{2}$$

where in equation (2) it is noted that $h\,\mathrm{d}h/\mathrm{d}x = -17.5$.

CVBEM TECHNIQUE

A complete development of the CVBEM numerical technique is provided by Hromadka and Guymon,[1] and in Hromadka.[5] In the following is presented a brief development of the CVBEM model using a constant trial function on each boundary element.

Let $\omega(z)$ be analytic in domain Ω and on boundary Γ (this implies that $\omega(z)$ is analytic in an open domain Ω^*

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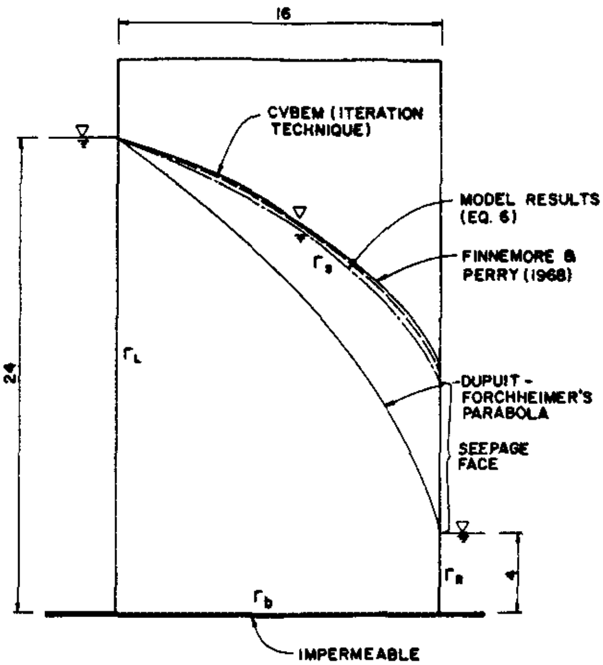

Figure 1. Study problem and modeling results

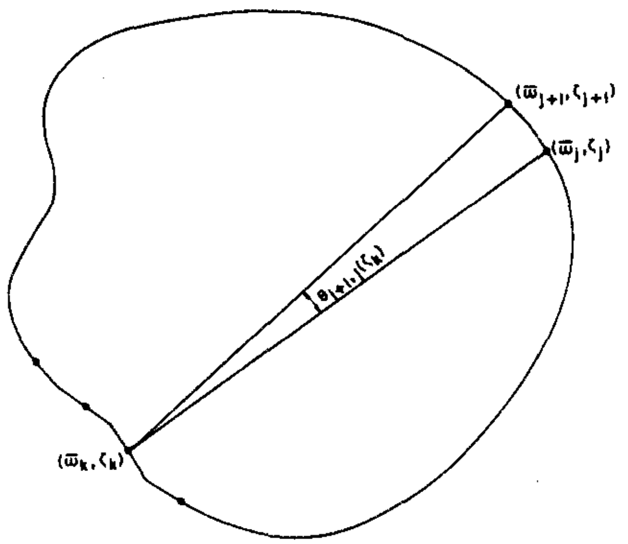

Figure 2. CVBEM geometrics

where $\Omega \cup \Gamma$ is a proper subset of Ω^*). Then by Cauchy's theorem:

$$\omega(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\omega(\zeta)\, d\zeta}{\zeta - z}; \quad z \in \Omega,\ z \notin \Gamma$$

The CVBEM develops an approximator function $\hat{\omega}(z)$ by substituting an approximation for $\omega(\zeta)$ in the integral. Let Γ be subdivided into m boundary elements Γ_j, $j = 1, 2, \ldots, m$. Letting $\omega(\zeta)$ be approximated by an element central value $\bar{\omega}_j$ in each Γ_j gives:

$$2\pi i\, \hat{\omega}(z) = \sum_{j=1}^{m} \bar{\omega}_j \int_{\Gamma_j} \frac{d\zeta}{\zeta - z} = \sum_{j=1}^{m} \bar{\omega}_j H_j(z) \qquad (3)$$

where $H_j(z)$ is the solution of the element integral as a function of point $z \in \Gamma$. Since $\bar{\omega}_j$ is specified at the midpoint ζ_j of Γ_j, the limiting value of $\hat{\omega}(\zeta_j)$ is determined by letting point $z \in \Omega$ approach ζ_j.[1] Collocating $\hat{\omega}(\zeta_j)$ to $\bar{\omega}_j$ at each ζ_j gives:

$$\pi i \bar{\omega}_k = \sum_{j \neq k} \bar{\omega}_j H_j(\zeta_k) \qquad (4)$$

where $\bar{\omega}_k$ (and $\bar{\omega}_j$) are nodal values defined by:

$$\bar{\omega}_k = \bar{\phi}_k + i\bar{\psi}_k$$

and $H_j(\zeta_k)$ is defined from Fig. 2 by:

$$H_j(\zeta_k) = \ln(|z_{j+1} - \zeta_k| / |z_j - \zeta_k|) + i\theta_{j+1,j}(\zeta_k) \qquad (5)$$

In equation (4), the bar notation indicates that the values are averaged over the element.

Due to the boundary conditions of the problem, the boundaries Γ_L and Γ_R are completely defined. On Γ_S, however, y (and ϕ) need to be determined; and on Γ_b, ϕ needs to be evaluated.

To solve for the unknown values, the complex logarithm function $H_j(\zeta_k)$ is expanded as:

$$H_j(\zeta_k + \delta y_j) = H_j(\zeta_k) + \frac{\partial H_j(\zeta_k)}{\partial y_j} \delta y_j \qquad (6)$$

The values for equation (6) depend on whether the terms of equation (3) include the varying phreatic surface of Γ_S. That is, the phreatic surface is estimated as:

$$\hat{y}(x) = h(x) + \delta y(x) \qquad (7)$$

where $\hat{y}(x)$ is the phreatic surface and $h(x)$ is defined by equation (2). In equations (6) and (7), δy_j, $\delta y(x)$, and δy all refer to displacements in the y-coordinate at node j, along the free water surface as a function of x, or at an arbitrary location, respectively. Substituting equations (6) and (7) into equation (3) results in a system of m equations of m unknowns, with some of the unknown terms combined as products of the form $\phi\delta y$. It is seen from equation (7) that the Dupuit solution to the flow problem is used as a boundary condition for the CVBEM model.

MODELING APPROACH

To evaluate the terms of (6), the computer program first evaluates the change in each $H_j(\zeta_k)$ with respect to a small increase in the y-coordinate for the nodes on Γ_S. Next, in order to simplify the products of the unknown variables, it is assumed that:

(1) the variation of $H_j(z)$ is negligible between nodes on Γ_S; and
(2) for products of $\phi\delta y$, it is assumed that $\phi = h(x)$ for nodes on Γ_S; and ϕ is linear for nodes on Γ_b.

The resulting matrix system is readily solvable by the usual matrix solution methods (e.g. Gaussian elimination). Other assumptions may be used in the above simplifications; however, for the type of problem considered, the above simplifications were found to be satisfactory.

DISCUSSION OF RESULTS

The results from applying the simple model are shown in Fig. 1. From the figure, the estimated phreatic surface lies between $h(x)$ and $\hat{y}(x)$ where $\hat{y}(x)$ is determined from a

20-node CVBEM iteration solution. Also shown is the domain solution of Finnemore and Perry.[3]

Generally, the iteration approach is to average the values of estimated $\phi(x)$ and $y(x)$ on Γ_S until the variation is negligible. However, with the approach presented in this note, a good estimate may be obtained by solving a single matrix which is a function of boundary coordinates.

CONCLUSIONS

The classic unknown phreatic surface seepage problem is approximated by a CVBEM approach which solves directly for the position of the phreatic surface without the use of an iteration scheme. In the phreatic surface problem considered, good results were obtained by this technique. Further research is needed, however, in the evaluation of limitations of the approach, and the possibility of expanding the technique for solving interface problems such as freshwater-seawater boundaries, or various other boundary condition problems involving the free water surface.

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