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REDUCING RELATIVE ERROR FROM THE CVBEM BY PROPER TREATMENT OF THE KNOWN BOUNDARY CONDITIONS

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SUMMARY

By a proper treatment of the known boundary conditions of a boundary value problem, a complex variable boundary element method (CVBEM) can be used to exactly satisfy the known nodal point boundary values. In this fashion, a numerical model can be developed which generates relative error information along the problem boundary that can be used to reduce the modelling error by either an integrated measure or a maximum relative error measure.

INTRODUCTION

Recently, the use of complex variable analytic function theory has been shown to be a powerful means of numerical analysis for approximating the Dirichlet, Neuman and mixed boundary value problems of the Laplace equation. Using a linear trial function approximator, Hunt and Isaacs[1] and Hromadka and Guymon[2] have analysed several two-dimensional boundary value problems. In a subsequent paper, Hromadka and Guymon[3] generalized the method into a complex variable boundary element method (CVBEM) which has a direct analogy to the well-known real variable boundary element method.

During the past several months, three papers were prepared on topics addressing the approximation error associated to the CVBEM. These three papers all deal with a complicated numerical method which promises to be a very useful tool for engineering analysis. Specifically, the CVBEM allows a direct evaluation of approximation error. Consequently, various techniques for minimizing error, nodal point placement and discretization of the boundary, and error magnitude estimation are available.

The three subject papers each address a specific topic involving the error analysis of the CVBEM. Hromadka and Guymon[4] deal with a method to strategically place nodal points on the boundary as prescribed by a relative error plot generated by a CVBEM computer program. This 'algorithm' is then specifically applied to soil freezing problems where a moving boundary freezing front is modelled. Hromadka[5] addresses the calculation of error magnitudes and the magnitudes of error produced by the CVBEM. Knowing such error magnitudes allows the engineer to place additional nodal points only where nodal points are needed to reduce error. Using the derived formulae, nodal points can actually be eliminated along the problem boundary, where error bounds are determined to be small. Finally, the current paper addresses a new and sophisticated approach to minimizing the CVBEM modelling error by setting a dual set of mathematic

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conditions: (a) matching nodal point boundary condition values, and (b) forcing the approximator function to match the unknown nodal point values imposing only one of the above two conditions. To reduce matrix computation effort, an iteration scheme is advanced which retains the smaller matrix requirements.

BOUNDARY INTEGRAL EQUATION FORMULATION

Consider a simply connected domain, Ω, with a simple closed contour boundary, Γ, as shown in Figure 1. The boundary can be subdivided into m boundary elements, Γ_j, such that

$$\Gamma = \bigcup_{j=1}^{m} \Gamma_j \tag{1}$$

On each boundary element, define two nodal points located at the element end points; for element j, the co-ordinates of the nodes are z_j and z_{j+1}. A simple linear trial function, $\alpha(s)$, is assumed on each element such that

$$\alpha(s) = \bar{\omega}_j s + \bar{\omega}_{j+1}(1-s);\ 0 \leqslant s \leqslant 1 \tag{2}$$

where $\bar{\omega}_j$ is the complex nodal values for node j, and where $\bar{\omega}_j = \bar{\phi}_j + i\bar{\psi}_j$. In equation (2), $\bar{\phi}_j$ and $\bar{\psi}_j$ are state variable and stream function nodal values at co-ordinate z_j. The bar notation signifies a specified nodal value.

The CVBEM utilizes an integral function $\hat{\omega}(z)$ defined by

$$2\pi i\hat{\omega}(z) = \sum_{j=1}^{m} \int_{\Gamma_j} \left(\frac{\alpha(\zeta)\mathrm{d}\zeta}{\zeta - z} \right)_j,\ z \in \Omega,\ z \notin \Gamma \tag{3}$$

where ζ is the complex variable of integration, $\alpha(\zeta)$ are the continuous trial functions, and subscript j refers to element contour Γ_j. Because the $\alpha(\zeta)$ are continuous on Γ_j, the approximation function $\hat{\omega}(z)$ is analytic for all z interior of Γ.

Equation (3) can be solved for any point z interior of Γ by noting that[1]

$$\begin{aligned} \int_{\Gamma_j} \frac{\alpha(\zeta)\mathrm{d}\zeta}{\zeta - z} = \bar{\omega}_{j+1}\left[1 + \left(\frac{z - z_j}{z_{j+1} - z_j} \right) H_j \right] \\ - \bar{\omega}_j \left[1 + \left(\frac{z - z_{j+1}}{z_{j+1} - z_j} \right) H_j \right] \end{aligned} \tag{4}$$

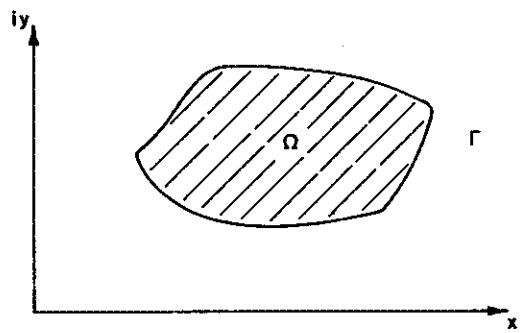

Figure 1. Problem domain, Ω, with boundary, Γ

where

$$H_j = \ln\left(\frac{z_{j+1} - z}{z_j - z}\right)$$

A boundary integral equation can be formulated for each nodal point by

$$2\pi i\hat{\omega}(z_1) = \lim_{z \to z_1^-} \sum_{j=1}^{m} \int_{\Gamma_j} \left(\frac{\alpha(\zeta)\mathrm{d}\zeta}{\zeta - z}\right)_j \tag{5}$$

where the limit is evaluated as z approaches arbitrary nodal co-ordinate z_1 from the interior of Γ. Solving equation (5) gives

$$2\pi i\hat{\omega}(z_1) = \bar{\omega}_1 H_1 + \sum_{j=2}^{m-1}\left[\bar{\omega}_{j+1}\left(\frac{z_1 - z_j}{z_{j+1} - z_j}\right) - \bar{\omega}_j\left(\frac{z_1 - z_{j+1}}{z_{j+1} - z_j}\right)\right] H_j \tag{6}$$

where

$$H_j = \ln\left(\frac{d(j+1,1)}{d(j,1)}\right) + i\theta(j+1,j) \tag{7}$$

and

$$H_1 = \ln\left(\frac{d(2,1)}{d(m,1)}\right) + i\theta(2,m)$$

In equation (7), $d(j+1,1)$ is the distance between nodal co-ordinates z_{j+1} and z_1, and $\theta(j+1,j)$ is the angle between co-ordinates z_{j+1} and z_j with vertex at z_1 (Figure 2). Additionally, $\theta(2,m)$ is the vertex exterior angle shown in Figure 2.

An examination of the approximation function definition of equation (6) reveals that $\hat{\omega}(z_1)$ is a function of the boundary geometry and nodal values, $\bar{\omega}_j$. Should the assumed trial functions $\alpha(\zeta)$ be

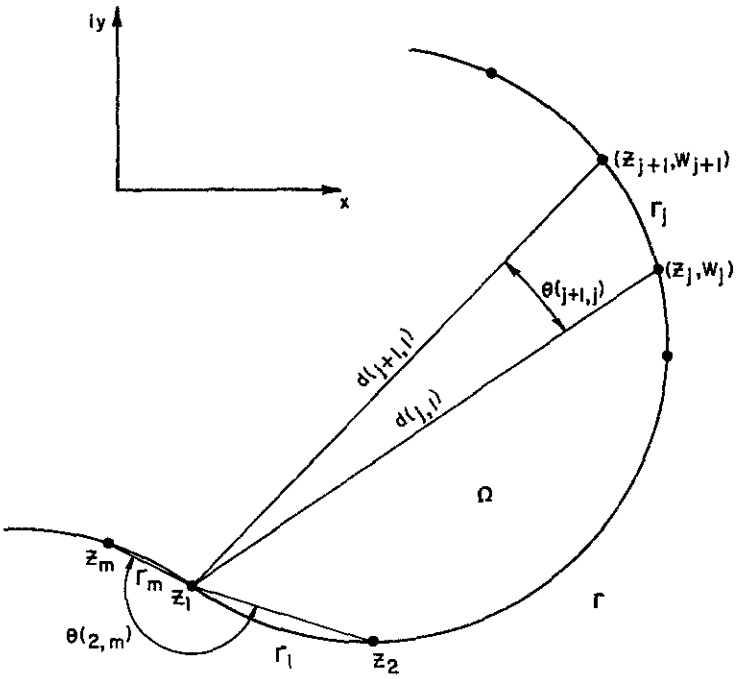

Figure 2. CVBEM linear trial function geometry

the solution of the boundary value problem, then $\hat{\omega}(z)$ is the solution of the boundary value problem and $\hat{\omega}(z_j) = \bar{\omega}(z_j)$, $j = 1, 2, \ldots, m$. Generally, however, $\hat{\omega}(z)$ is not the desired solution of $\omega(z) = \phi + i\psi$, and $\hat{\omega}(z_j) \neq \bar{\omega}(z_j)$.

MODEL DEVELOPMENT

The nodal values, $\bar{\omega}_j$, are composed of two components $\bar{\omega}_j = \bar{\phi}_j + i\bar{\psi}_j$, where either $\bar{\phi}$ or $\bar{\psi}$ is known at each z_j by the given boundary condition definitions. Consequently, each nodal point has an assigned known boundary value and a corresponding unknown boundary value. Should both boundary nodal values be known at each z_j, then the approximation function $\hat{\omega}(z)$ is defined throughout the interior of Γ. Therefore, in order to calculate $\hat{\omega}(z_j)$ values, estimates of the unknown nodal boundary condition values are required. In the following discussion, it is assumed that $\bar{\phi}_j$ is specified at each $z_j (\bar{\phi}_j = \phi_j)$ and the $\bar{\psi}_j$ are unknown (except for a single nodal point value where the constant of integration iC is evaluated). The discussion is assumed to be extendable to the case of mixed boundary conditions. The following notation is used for the three sets of nodal point values:

$\omega_j = \omega(z_j) = \phi_j + i\psi_j$; exact solution of boundary value problem solution at node j
$\bar{\omega}_j = \bar{\phi}_j + i\bar{\psi}_j$; boundary condition nodal values
$\hat{\omega}_j = \hat{\phi}_j + i\hat{\psi}_j$; approximation values at node j

Solution of equation (6) for each nodal point results in m linear equations which can be written in matrix form as

$$\hat{\boldsymbol{\omega}} = \mathbf{C}_R(\bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\psi}}) + i\mathbf{C}_I(\bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\psi}}) \tag{8}$$

where $\mathbf{C}_R$ and $\mathbf{C}_I$ are $m \times 2m$ matrices of real constants representing the real and imaginary parts of the boundary integral equations, respectively. From (8), two matrix systems require simultaneous solution,

$$\begin{aligned} \hat{\boldsymbol{\phi}} &= \mathbf{C}_R(\bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\psi}}) \\ \hat{\boldsymbol{\psi}} &= \mathbf{C}_I(\bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\psi}}) \end{aligned} \tag{9}$$

where $(\bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\psi}})$ is the array of nodal point boundary values.

One method of solving equation (9) is to set $\hat{\boldsymbol{\psi}} = \bar{\boldsymbol{\psi}}$ (References 1 and 2) and solve

$$\bar{\boldsymbol{\psi}} = \mathbf{C}_I(\bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\psi}}) \tag{10}$$

using the known $\bar{\boldsymbol{\phi}}$, followed by the solution of $\hat{\boldsymbol{\phi}} = \mathbf{C}_R(\bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\psi}})$, which results in values for $\bar{\boldsymbol{\psi}}$ such that $\bar{\boldsymbol{\psi}} = \bar{\boldsymbol{\psi}}$, but generally, $\hat{\boldsymbol{\phi}} \neq \bar{\boldsymbol{\phi}}$. It should be noted that the calculation of $\bar{\boldsymbol{\psi}}$ can also be achieved by solving

$$\bar{\boldsymbol{\phi}} = \mathbf{C}_R(\bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\psi}}) \tag{11}$$

which results in $\hat{\boldsymbol{\phi}} = \bar{\boldsymbol{\phi}}$ but generally, $\hat{\boldsymbol{\psi}} \neq \bar{\boldsymbol{\psi}}$. One may use equation (11) due to the approximator $\hat{\omega}(z)$ matching the boundary condition values at each nodal point, and then evaluate the relative error of $(\omega - \hat{\omega})$ by analysis of $(\bar{\psi} - \hat{\psi})$ and $(\phi - \hat{\phi})$ on Γ. However, $(\bar{\psi}, \hat{\psi})$ is not known continuously on Γ. Should the model of equation (10) be used, evaluation of the relative error of $(\bar{\psi}, \hat{\psi})$ is aided by $\bar{\psi} = \hat{\psi}$, and $(\phi - \hat{\phi})$ is known continuously on Γ.

Hromadka and Guymon[4] use the model of equation (10) to develop values of $\bar{\psi}$ and then define an analytic relative error function $e(z) = \omega(z) - \hat{\omega}(z)$ which is evaluated on Γ. From that study, integrated relative error bounds are derived and an algorithm advanced to reduce relative error by evaluating the continuously known boundary condition approximation relative error of $(\phi - \hat{\phi})$.

The algorithm requires the addition of nodal points to the boundary in regions where computed relative error is relatively large.

In order to better satisfy the boundary conditions and improve the numerical algorithm, a different scheme is presented in the following paragraphs. The scheme requires the approximation function $\hat{\omega}(z)$ to equal the known boundary nodal values (i.e. $\hat{\boldsymbol{\phi}} = \bar{\boldsymbol{\phi}}$) and also require that $\hat{\boldsymbol{\psi}} = \bar{\boldsymbol{\psi}}$ in order to preserve the more desirable limitations on the unknown nodal point boundary condition values. This approach can be accommodated by redefining the known nodal boundary condition values ($\bar{\phi}_j = \phi(z_j)$) to new values, $\bar{\phi}_j = \phi_j^*$, such that

$$\hat{\boldsymbol{\phi}}_j = \{\phi(z_j)\} = \mathbf{C}_R(\boldsymbol{\phi}_j^*, \bar{\boldsymbol{\psi}}_j) \tag{12}$$

and

$$\bar{\boldsymbol{\psi}}_j = \mathbf{C}_I(\boldsymbol{\phi}_j^*, \bar{\boldsymbol{\psi}}_j)$$

The above process requires that new values ϕ_j^* be computed at each nodal point such that the approximation function $\hat{\omega}(z)$ has the property that

$$\hat{\omega}(z_j) = \hat{\phi}_j + i\hat{\psi}_j = \phi(z_j) + i\bar{\psi}_j, j = 1, 2, \ldots, m \tag{13}$$

ITERATION SCHEME

A major difficulty in imposing the double requirements on $\hat{\omega}(z)$ as defined by equation (12) and (13) is that a $2m \times 2m$ matrix system results which is 4 times the sizes of the $m \times m$ system determined by the simpler approaches of equations (10) or (11).

However, by using another approach, namely by using an iteration procedure as shown below, the system of equations (12) and (13) can be solved by an $m \times m$ matrix system. Using the model of equation (10), the first step of the iteration procedure is to set

$$\bar{\boldsymbol{\psi}} = \mathbf{B}_I\bar{\boldsymbol{\psi}} + \mathbf{L}_I\bar{\boldsymbol{\phi}} \tag{14}$$

where $\mathbf{B}_I$ and $\mathbf{L}_I$ are $m \times m$ matrices resulting from the global matrix of equation (10), $\mathbf{C}_I$. Because $\bar{\boldsymbol{\phi}}$ is known ($\bar{\boldsymbol{\phi}} = \{\phi(z_j)\}$), solution of equation (14) results in an estimate of $\bar{\boldsymbol{\psi}}$. Using the estimated $\bar{\boldsymbol{\psi}}$, values for $\hat{\boldsymbol{\phi}}$ are computed by

$$\hat{\boldsymbol{\phi}} = \mathbf{B}_R\bar{\boldsymbol{\psi}} + \mathbf{L}_R\bar{\boldsymbol{\phi}} \tag{15}$$

where $\mathbf{B}_R$ and $\mathbf{L}_R$ are $m \times m$ matrices from global matrix $\mathbf{C}_R$. From equation (14), and dropping matrix notation,

$$\hat{\psi} = \bar{\psi} = (I - B_I)^{-1}L_I\bar{\phi} \tag{16}$$

where I is the $m \times m$ identity matrix. Then, in simpler notation,

$$\begin{aligned} \hat{\phi} &= D\bar{\phi} \\ \hat{\psi} &= D^*\bar{\phi} \end{aligned} \tag{17}$$

where

$$\begin{aligned} D^* &= (I - B_I)^{-1}L_I \\ D &= B_R(I - B_I)^{-1}L_I + L_R \end{aligned} \tag{18}$$

A known boundary condition relative error, $\phi_e = \phi - \hat{\phi}$, can be computed continuously along Γ. The nodal values of ϕ_e are given by

$$\phi_e = \bar{\phi} - \hat{\phi} \tag{19}$$

The ϕ_e (and hence $\bar{\phi}_e$) function is known continuously on Γ and can be used to locate additional

nodal points on Γ to locally reduce error. However, in the current development the nodal point structure is assumed fixed. Using ϕ_e as the boundary condition for a new boundary value problem, an approximation function $\hat{e}(z)$ can be determined analogous to the development of $\hat{\omega}(z)$. Letting $e(z) = \phi_e + i\psi_e$, where $\psi_e = \psi - \bar{\psi}$, the second step of the iteration process is setting

$$\begin{aligned} \hat{\bar{\psi}}_e &= \bar{\bar{\psi}}_e = B_I\bar{\psi}_e + L_I\bar{\phi}_e \\ \hat{\bar{\phi}}_e &= B_R\bar{\psi}_e + L_R\bar{\phi}_e \end{aligned} \tag{20}$$

where $\bar{\phi}_e$ and $\bar{\psi}_e$ are nodal point values of the harmonic ϕ_e and ψ_e functions. In simpler notation, equation (20) can be written as

$$\begin{aligned} \hat{\bar{\psi}}_e &= \bar{\bar{\psi}}_e = D^*(I - D)\bar{\phi} \\ \hat{\bar{\phi}}_e &= (B_R D^* + L_R)(I - D)\bar{\phi} = D(I - D)\bar{\phi} \end{aligned} \tag{21}$$

in which D and D^* are as defined before. The next relative error function is defined by

$$e'(z) = e(z) - \hat{e}(z) \tag{22}$$

Repeating the above process, the several approximation functions can be summed giving

$$\begin{aligned} \phi_A &= \hat{\phi} + \hat{\phi}_e + \hat{\phi}'_e + \cdots \\ \psi_A &= \hat{\psi} + \hat{\psi}_e + \hat{\psi}'_e \cdots \end{aligned} \tag{23}$$

or, in simpler notation,

$$\begin{aligned} \phi_A &= [I + (I - D) + (I - D)^2 + \cdots]D\bar{\phi} \\ \psi_A &= [I + (I - D) + (I - D)^2 + \cdots]D^*\bar{\phi} \end{aligned} \tag{24}$$

where ϕ_A and ψ_A are the summed nodal point approximation values.

By examination, the two infinite series of equation (24) are not unconditionally convergent. Consequently, either the global matrix D needs to be verified that equation (24) converges, or the incremental error contributions need to be examined to ensure that convergence is occurring. In

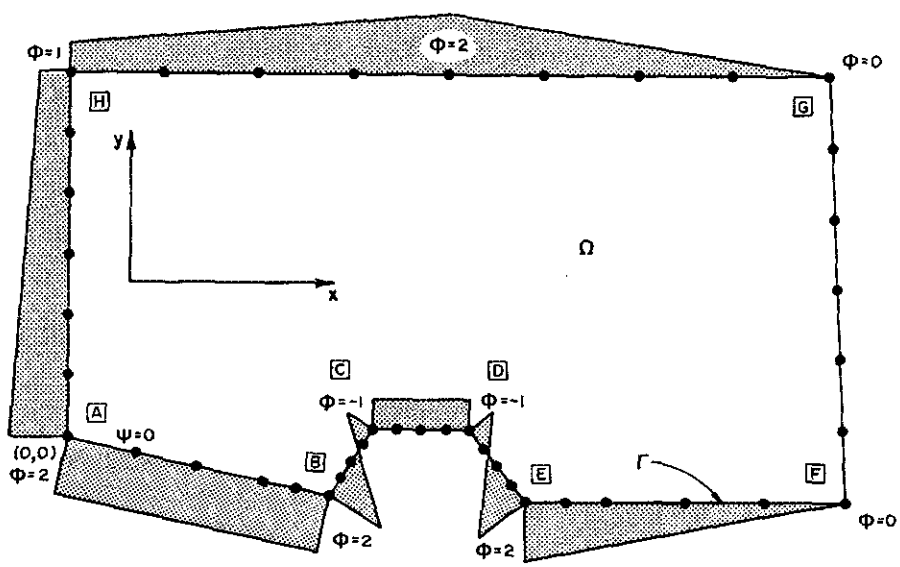

Figure 3. Test problem definition showing iteration approach nodal placement and specified boundary conditions

practice, usually only one or two iterations are used to find the required boundary values as the incremental error contributions generally become negligible. However, in cases where convergence is not occurring, additional nodal points need to be added to the boundary in order to reduce the magnitudes of the incremental error contributions.

APPLICATIONS

As an application of the method, a difficult boundary value problem was numerically approximated by both the iteration process of equation (24) and by the solution of equation (10) for the special case of doubling the number of boundary nodal points. Figure 3 shows the test boundary value problem geometry and imposed boundary conditions. In this test problem it is assumed that the state variable, ϕ, is known continuously along the boundary, Γ, and the objective is to evaluate the stream function, ψ, along Γ given that a reference value of $\psi = 0$ is assumed at the origin of the co-ordinate axis.

Figure 4 shows the computed relative error values of $(\phi - \bar{\phi})$ along Γ after three consecutive applications of equation (24). Also shown in Figure 4 is the relative error values of $(\phi - \hat{\phi})$ along Γ from the approximation of equation (10). From Figure 4, a comparable relative error distribution of $(\phi - \hat{\phi})$ is achieved by the iteration approach as obtained by a double nodal point density CVBEM model approximation.

DISCUSSION

The application problem illustrates the use of the CVBEM in developing an approximation function, $\hat{\omega}$, and also in determining an actual relative error distribution of $(\omega - \hat{\omega})$ along the problem boundary. Because the true solution of the boundary value problem $(\omega = \phi + i\psi)$ is partially given by the boundary conditions (e.g. ϕ), the task of the numerical analysis is to determine the unknown values (e.g. ψ) along Γ. The CVBEM develops the approximation $\hat{\omega} = \hat{\phi} + i\hat{\psi}$ which is based on the integral function of boundary values and geometry (equation 3). Consequently, we want the relative error function $e = \omega - \hat{\omega}$ to be identically zero along Γ.

Because we are dealing with analytic functions in $\Gamma^- \cup \Omega^-$, then the error function e is analytic

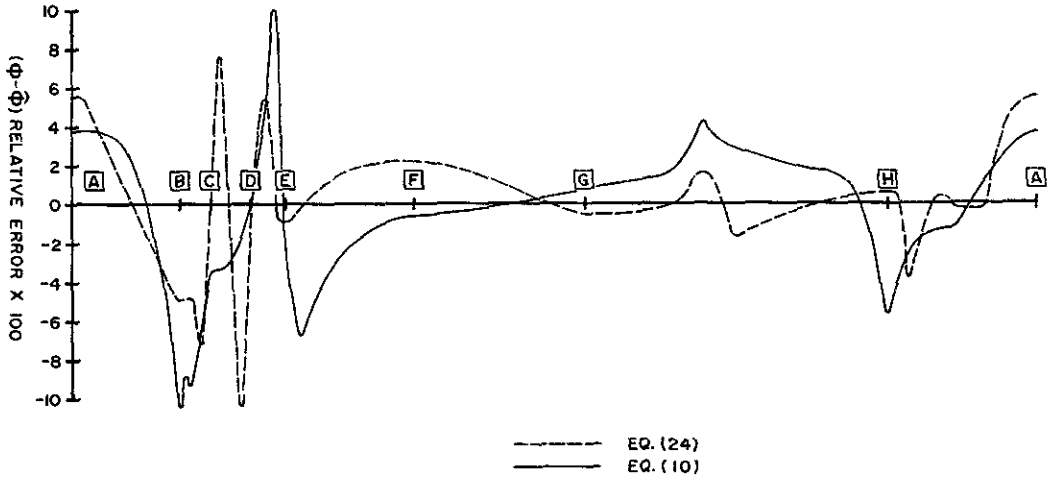

Figure 4. Comparison of boundary relative error values

and can be studied with respect to the Cauchy integral theorems. Namely,

$$\int_{\Gamma^-} e(\zeta)\,d\zeta = 0 \tag{25}$$

where Γ^- is simply connected contour arbitrarily close to but interior of Γ, and Ω^- is the interior of Γ^-. But e is composed of two harmonic functions ϕ_e and ψ_e such that

$$e = \phi_e + i\psi_e \tag{26}$$

where

$$\phi_e = \phi - \hat{\phi}$$
$$\psi_e = \psi - \hat{\psi}$$

For the subject study case, ϕ_e is known continuously on Γ^- (and Γ) and

$$-\int_{\Gamma^-} \phi_e\,dz = i\int_{\Gamma^-} \psi_e\,dz \tag{27}$$

For the mixed boundary condition case, Hromadka and Guymon[4] develop integral equations similar to equation (27). Since the CVBEM develops a matrix system as a function of specified $\bar{\phi}_j$ or $\bar{\psi}_j$ nodal values, the various possibilities for which type of nodal value is defined at each $z_j \in \Gamma$ still results in an approximation function $\hat{\omega}(z)$ which is analytic on Ω. Consequently, integrated relative error bounds exist for the CVBEM which can be actually evaluated for at least the function ϕ_e on Γ.

Further research is needed for several topics. For example, the sensitivity of the relative error to nodal point placement is of concern, especially when comparing the nodal point CVBEM error minimization to domain collocation methods. Another topic is the approximation bounds of the ψ_e error given a continuously defined ϕ_e function on Γ.

CONCLUSIONS

In this paper a new method of reducing the ϕ_e error function is proposed which entails the computation of a set of specified boundary conditions so that the $\hat{\phi}(z_j) = \phi(z_j), j = 1, 2, \ldots, m$. The resulting approximation function is shown to reduce relative error magnitudes similar to the effect of doubling the number of nodal points on the model boundary. It is concluded that the CVBEM is a very worthwhile tool for numerical analysis, due to the convenience of a relative error measure being a by-product of the modelling solution, and that this measure of error can be used to develop subsequent models which have even smaller magnitudes of relative error.

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