A Simple Model of Ice Segregation Using an Analytic Function to Model Heat and Soil-Water Flow

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For slowly moving freezing fronts in soil, the heat-transport equation may be approximated by the Laplacian of temperature. Consequently, potential theory may be assumed to apply and the temperature state can be approximated by an analytic function. The movement of freezing fronts may be approximated by a time-stepped solution of the phase-change problem, thus solving directly for heat flow across a freezing or thawing front. Moisture transport may approximated by using an exact solution of the moisture-transport equation assuming quasi-steady-state conditions, appropriate boundary conditions, and an exponential function relating unsaturated hydraulic conductivity (defined within the thawed zones) to pore water pressure (tension). This approach is used to develop a single model of ice segregation (frost-heave) in freezing soils. Applications to published and experimental one-dimension soil column freezing data show promising results.

Introduction

This paper reports on the development of a simple two-dimensional model of coupled heat and soil-water flow in freezing or thawing soil. The model also estimates ice-segregation (frost-heave) evolution. Ice segregation in soil results from water drawn into a freezing zone by hydraulic gradients created by the freezing of soil-water. Thus, with a favorable balance between the rate of heat extraction and the rate of water transport to a freezing zone, segregated ice lenses may form.

Predicting the rate of ice segregation in soil is an old problem that has received much recent attention by a number of investigators who have attempted to model ice segregation by means of mathematical models. O'Neill [8] reviews some of these efforts. Generally, most modeling efforts encompass modeling equations of coupled heat and moisture transport as well as a variety of ancillary equations that estimate parameters and attempt to model the complex physics and thermodynamic processes in freezing zones.

In contrast to these approximate models, a simple model of ice segregation such as discussed by Outcalt [9] or earlier by Arakawa [1], offers advantages over the multiparameter models advanced in the literature. First, a simple model is easily understandable to the practicing engineer, whereas a relatively complex model such as Guymon and others [3] and Harlan [4] limits its audience to a specialized few. Second, a complex model makes it more difficult to determine the source of modeling errors since there is often an unknown interaction between model parameters. The more parameters imbedded in a model causes greater difficulty in estimating the uncertainty. However, complex models are needed due to the nature of the frost-heave process, for which a simple model may fail to achieve the desired accuracy. Of course, just because a model is complex and multiparameter does not guarantee the model accuracy.

The primary objective of this paper is to expand on the one-dimensional model of ice segregation advanced by Outcalt [9]. The proposed model will accommodate two-dimensional problems and will be based on a simple thermodynamic balance of heat flow along a freezing front of differential thickness. A secondary objective of this paper is to introduce a numerical analog based on analytical function complex variable theory by fitting a polynomial of the complex variable potential and stream functions to the prescribed boundary conditions. The solution is determined by the values along the domain boundary analogous to the usual BIEM (boundary integral equation methods).

For ice segregation moving boundary problems where phase-change latent heat effects dominate the heat transport process, the heat-balance equations may, in some general cases, be approximated by the Laplace equation coupled with the boundary conditions modified to include the effects of phase change. Similar assumptions are made for the Stephan solution, which has been used for years to calculate the thickness of ice. To do this, the dynamic component of the classical heat-transport equation is assumed negligible when freezing or thawing a soil region. Moreover, it is necessary to assume an isotropic, homogeneous solution domain. However, by means of a suitable coordinate transformation for relatively geometrically simple regions, anisotropic or even heterogeneous domains may be transformed into a region in which potential theory may apply. For these types of

Contributed by the Arctic Committee of the Petroleum Division and presented at the 3rd International Symposium on Offshore Mechanics and Arctic Engineering, ETCE, New Orleans, La., February 12-16, 1984, of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS. Manuscript received by the Petroleum Division, November 1, 1983; revised manuscript received February [illegible], 1984.

problems, complex variable modeling techniques may be applied, which may reduce computer storage and execution times when compared to classical domain methods. Generally, in freezing problems we are interested primarily in the location of the freezing front and in the estimation of heat and soil-water flux values normal to the freezing front. The proposed model of ice segregation focuses on these two types of problems directly.

Discussion of Modeling Approach

The proposed simple model of ice segregation is based on the one-dimensional model developed in Outcalt [9] and extends the two-dimensional BIEM geothermal model of Hromadka and Guymon [5] to include soil-water flow. In the following, a general summary of the modeling assumptions used in the ice segregation model will be presented.

The model is applicable to a saturated or unsaturated soil which is subjected to constant or stepwise constant upper and lower boundary conditions of temperature and soil-water pore pressure. The coupling of boundary conditions to the modeling domain is restricted by the capability of the model to approximate a variation in boundary conditions by time-averaged steady-state solutions of the governing flow equations. This limitation of the model will become apparent after the following description of the model development. Major assumptions employed in the model are:

1 Unsaturated soil-water flow theory applies and the extended Darcy's law is valid in the unfrozen soil. Moisture movement is driven by the total hydraulic head energy gradient.

2 The classical heat equation applies to the entire soil system.

3 Soil-water phase change latent heat effects dominate the heat-flow equation and the transient heat and convention terms can be considered negligible. This assumption may be acceptable for problems involving a slow freezing/thawing of fine-grained soils such as silts. Frost-susceptible soils where ice segregation is most likely to occur favors this assumption in that the freezing front propagation is slow. (This assumption may fail for high ice segregation ratio case studies.)

4 The volumetric latent heat of fusion, L, is constant in the temperature ranges found in seasonally freezing/thawing soils.

5 Ice segregation occurs when moisture drawn into the freezing front exceeds the soil porosity less the unfrozen water content.

6 Hysteresis is not present and all functions are single-valued and piecewise continuous to approximate possible jump discontinuities.

7 Soil-water salt-transport effects are negligible. The freezing front maintains a constant temperature, such as 0°C.

8 Overburden and surcharge effects are presently neglected.

9 The freezing front separates the problem domain into completely frozen and completely unfrozen regions.

10 Soil-water flow in the frozen regions is negligible. This assumption may be acceptable for problems involving a seasonally freezing/thawing soil where freezing occurs for only a few months and not for long durations, as would be imposed by a gas pipeline operated continuously at subfreezing temperatures.

11 The only soil deformation considered is due to ice segregation and this deformation is lumped vertically above the freezing front.

12 The problem domain is homogeneous and isotropic. Nonhomogeneous and anisotropic domains can be rescaled (for simple cases) into another homogeneous, isotropic domain.

13 The heat and soil-water flow equations can be modeled as quasi-steady-state processes for small durations of time. All time-dependent state variable (dynamic) terms can be assumed negligible compared to the dominating phase-change terms.

The heat flow PDE (partial differential equation) can be modeled as the simple Laplace relation defined by

$$k_f \nabla^2 T(x,y) = 0\,;\ (x,y) \in \Omega_f$$
$$k_t \nabla^2 T(x,y) = 0\,;\ (x,y) \in \Omega_t \tag{1}$$

where convected heat effects are assumed negligible; k_f,k_t are frozen and thawed thermal conductivities; Ω_f,Ω_t are frozen and thawed subregions of the global domain Ω; and T is temperature. It can be noted that the freezing-front contour, Γ^*, separates Ω_f and Ω_t and that any homogeneous effects due to k_f and k_t are isolated by the various subproblems defined in equation (1). Domain numerical models generally require global matrix regeneration due to nonlinear conduction parameters in a finite element or control volume; this step is eliminated by the proposed modeling approach. The freezing contour, Γ^*, is defined by

$$\Gamma^* \equiv \{ (x,y) : T(x,y) = 0^\circ\mathrm{C} \} \tag{2}$$

which geometrically represents the 0°C isotherm. Propagation of Γ^* in Ω is determined by a basic heat-balance relation

$$L \frac{ds}{dt} = \sum_{\Gamma^*} q_n \tag{3}$$

where ds/dt is a movement of coordinates on Γ^* due to the net heat evolution from the summed heat fluxes, q_n, normal to Γ^* with the sign convention defined according to ds/dt.

Soil-water flow is considered as vertical only (i.e., a one-dimensional model). Horizontal flow is assumed to be negligible. However, a large class of real world problems, such as roadway freezing problems, are capable of being modeled by this simple modeling approach. Soil-water flow is modeled in a two-step analog. First, the soil system is discretized into vertical finite element strips wherein a background steady-state water content (or pore water pressure) profile is determined for each strip as a function of the strip's current boundary conditions (on the top and bottom). The soil-water flow-conduction parameter, $D(\theta)$, is assumed to be a simple exponential function such that a steady-state moisture flux is readily computed (neglecting gravity effects in the soil-water flow PDE) along the strip boundary

$$D(\theta) = ae^{b\theta} \tag{4}$$

The second process is a soil column dewatering model for each vertical finite element strip. In this second analog, soil-water flux is approximated along the finite element strip until the background steady-state water-content profile is reached. Figure 1 illustrates these two models used in the total soil-water flow analog.

The foregoing model assumptions, although restrictive to a total ice segregation model, may be generally attractive for use on problems involving a seasonally freezing/thawing fine-grained soil which is frost suceptible. Special boundary considerations of geometry and soil-water flow can be easily addressed on a problem-by-problem basis. Some of the model's advantages are as follows:

1 The model is based on a simple approach for estimating ice segregation, which accounts for heat and soil-water transport.

2 The freezing front and frost-heave development are defined directly without a domain mesh regeneration.

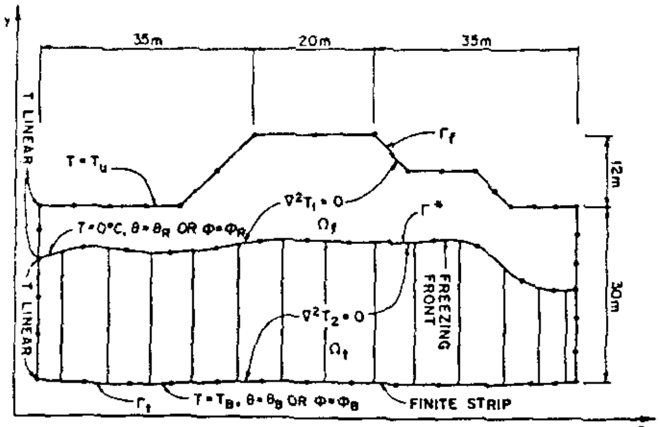

Fig. 1 Example embankment showing boundary nodal placement for heat-transfer equation and finite strips for moisture-transfer equation

3 Nonlinearity of conduction parameters due to phase change is estimated.

4 For homogeneous problems, fewer nodal points are required in this model than in domain models.

5 Computer coding requirements are significantly reduced over numerical models using domain methods.

6 The model requires fewer parameters than domain-method models which incorporate dynamic terms.

7 The application problem provides an indication as to the success (or failure) of a model developed from these simplifying assumptions.

Numerical Model Formulation

Soil-water flow in unfrozen soil is modeled by means of several quasi-one-dimensional submodels of soil-water flow defined in a vertical strip discretization of the soil matrix located below the freezing front, Γ^* (Fig. 1). In the two-dimensional model, gravitational effects are neglected in the governing PDE and a simple soil-water diffusivity model is used in each vertical strip

$$\frac{\partial}{\partial y} D(\theta) \frac{\partial \theta}{\partial y} = 0 \tag{5}$$

where the soil-water diffusivity, D, is assumed a function of volumetric water content as given in equation (4). In each vertical strip, an upper and lower boundary condition is specified according to

$$\theta = \theta_0,\ y = 0 \text{ (water table)} \tag{6a}$$

$$\theta = \theta_u,\ y = H(\Gamma^*) \tag{6b}$$

where condition equation (6a) reflects a saturated soil (θ_0 equals the soil porosity) at the water table, and in equation (6b) θ_u is an unfrozen water content characteristic to the soil (Guymon and others, [3]). Outcalt [9] assumes θ is linear between Γ^* and the water table (which is separated by length H). In the two-dimensional model, equation (5) is integrated to directly calculate a steady-state soil-water flux, v, from the water table to Γ^* giving (for a and b constants)

$$v \equiv \frac{a}{bH}\left(e^{b\theta_0} - e^{b\theta_u}\right) \tag{7}$$

In the limit as the exponent term b approaches zero, equation (7) approaches the simple linear gradient model used by Outcalt [9]

$$\lim_{b \to 0} v = \frac{a}{H}\left(\theta_0 - \theta_u\right) \tag{8}$$

Equation (8) is used to dewater the soil column in each vertical strip until the specified initial condition soil-water content profile equals the minimum steady-state water content profile determined from equations (5) and (6). After the necessary dewatering of the vertical strip, equation (7) is then used as the minimum value of soil-water flux feeding the slowly moving freezing front, Γ^*. In the Outcalt model, an "apparent" hydraulic conductivity is required for use of equation (8) in the unfrozen zone; this calculation is not necessary for the model of equation (7).

Analogous to the soil-water flow model, the freezing front propagation is assumed to be slow enough to justify the elimination of the dynamic heat capacitance term from the classical heat equation. This allows the calculation of heat flux, q_n, along the freezing front, Γ^*, to be accomplished by using a steady-state temperature profile, equation (1), within the problem domain, Ω.

Figure 1 shows an example solution domain. A constant temperature is specified for T_u and T_L (where T is the potential function) with the sides of the roadway embankment problem being specified with values of $Q = Q_L$ and Q_R (where Q is a stream function). Neumann boundary conditions can be used on the left and right sides in determining Q_L and Q_R, or an equivalent T_L and T_R. Any of the usual boundary integral approaches can be used for this problem; a complex polynomial approximation is used in this model due to the significant reduction in computational effort when compared to other BIEM requirements.

Assuming the freezing front location to be defined at some time t_0, the dynamic heat evolution problem is approximated by solving the Laplace relations (Fig. 1) to estimate the heat-flux values along the freezing front during a timestep, Δt. For example, in the problem studied, timesteps of one day are used with good results. From the estimated heat-flux values, the change in the freezing front is calculated from equation (3). That is, a method to calculate the change in the freezing front coordinates is to calculate the change in the nodal point coordinates in the direction of net normal heat flux. For nodal points located at the midpoint of boundary elements, the determination of new coordinates at the freezing front may be estimated by a simple balance between the volume of soil frozen and the time-integrated heat evolved. Due to the model's basic assumption of phase-change effects dominating the entire heat-transportation process, the freezing front evolution is slow and the simple freezing front evolution model was found to be adequate for the problems tested. The freezing front contour, Γ^* (Fig. 1), separates an otherwise simply connected domain Ω into a frozen and thawed subdomain, Ω_f and Ω_t, respectively. Among any contour C the steady-state thermal condition assumed in Ω (for small durations of time) implies that

$$\int_C Q_n ds = \int_C Q_t ds = 0 \tag{9}$$

where Q_n, Q_t are normal and tangential components of the heat flux along the contour C; and ds is a differential arc-length. Equation (9) establishes that the temperature function, or state variable T, is harmonic and satisfies the Laplace equation

$$\nabla^2 T = 0;\ T \epsilon \Omega_f, \Omega_t \tag{10}$$

The harmonic conjugate stream function Q exists in Ω, and Ω_t and is related to T by the Cauchy-Reimann equations of complex variable theory (Churchill [2])

$$\frac{\partial T}{\partial x} = \frac{\partial Q}{\partial y} \tag{11a}$$

$$\frac{\partial T}{\partial y} = \frac{-\partial Q}{\partial x} \tag{11b}$$

The complex temperature $\xi(z)$ is defined in each of Ω_f and Ω_t by

$$\xi_f(z) = T_f(x,y) + iQ_f(x,y);(x,y)\epsilon\Omega_f \tag{12a}$$

$$\xi_t(z) = T_t(x,y) + iQ_t(x,y);(x,y)\epsilon\Omega_t \tag{12b}$$

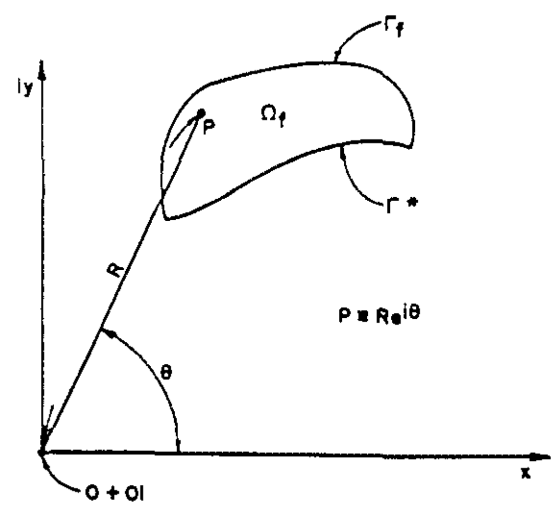

Fig. 2 Complex variable polar coordinates

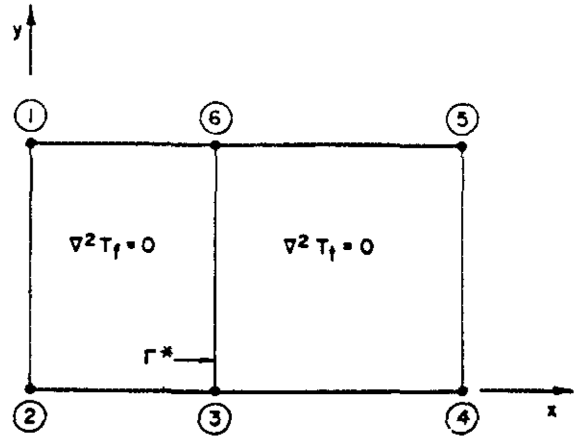

Fig. 3 Six node BIEM model of one-dimensional soil column

By definition, both $\xi_f(z)$ and $\xi_t(z)$ are analytic in their respective subdomains and can be expanded by appropriate Taylor-series expansions. As an approximation, a Taylor series will be developed, centered at the complex plane origin, which satisfies the specified boundary conditions for both $\xi_f(z)$ and $\xi_t(z)$, respectively. Letting $w(z)$ denote either $\xi_f(z)$ or $\xi_t(z)$, $w(z)$ is approximated in its appropriate subdomain by

$$w(z) = (\alpha_o + i\beta_o) + (\alpha_1 + i\beta_1)z + \ldots + (\alpha_n + i\beta_n)z^n \quad (13)$$

Using polar coordinates (Fig. 2), the Euler formula describes point z by

$$z = x + iy = R(\cos\theta + i\sin\theta) = \mathrm{Re}^{i\theta} \quad (14)$$

and by de Moivre's theorem

$$z^k = R^k e^{ik\theta},\ k = 0,1,2,\ \ldots \quad (15)$$

Combining equations (13) and (15), expansions for both $T(z)$ and $Q(z)$ of the analytic function $w(z)$ is given for some point in Ω_f or Ω_t

$$\begin{aligned} T = {} & \alpha_0 + \alpha_1 R\cos\theta - \beta_1 R\sin\theta \\ & + \alpha_2 R^2\cos 2\theta - \beta_2 R^2 \sin 2\theta \\ & + \ldots \\ & + \alpha_n R^n \cos n\theta - \beta_n R^n \sin n\theta \end{aligned} \quad (16a)$$

$$\begin{aligned} Q = {} & \beta_0 + \beta_1 R\cos\theta + \alpha_1 R\sin\theta \\ & + \beta_2 R^2\cos 2\theta + \alpha_2 R^2 \sin 2\theta \\ & + \ldots \\ & + \beta_n R^n \cos n\theta + \alpha_n R^n \sin n\theta \end{aligned} \quad (16b)$$

where in equation (16), a $(k-1)$ order complex polynomial can be determined given $2k$ values of T or Q for each appropriate subregion.

In the ice segregation model, values of either T or Q are known on the contours Γ^*, Γ_t, and Γ_f. Consequently, a nodal point discretization of the boundary of Γ_f and Γ^* with specified values of T or Q at each node can be used to develop the (α_i, β_i) values of the $\xi_f(z)$ polynomial; and similarly, the same holds for the $\xi_t(z)$ polynomial. Specifically, the same nodal point discretization of Γ^* is used for both the determination of $\xi_t(z)$ and $\xi_f(z)$, and values of T and heat flux, q_n, are determined on Γ^* by the Cauchy-Riemann relations

$$T \equiv 0^\circ\mathrm{C},\ \ (x,y)\epsilon\Gamma^* \quad (17a)$$

$$\frac{\partial T}{\partial n} = \frac{\partial Q}{\partial x},\ \ (x,y)\epsilon\Gamma^* \quad (17b)$$

where equation (17b) holds true due to Γ^* being an isothermal, and where the sign of the gradient depends on the direction of the contour tangential with respect to Ω_f or Ω_t, with n being the outward normal direction.

In subdomain Ω_f, a global matrix is developed from equation (16) based on specified values of either T or Q along Γ_f and Γ^*

$$\begin{bmatrix} T_1 \\ Q_1 \\ T_2 \\ Q_2 \\ \cdot \\ \cdot \\ \cdot \\ T_{2k} \\ Q_{2k} \end{bmatrix} = \begin{bmatrix} 1 & 0 & R_1\cos\theta_1 & -R_1\sin\theta_1 & \ldots & \alpha_o \\ 0 & 1 & R_1\sin\theta_1 & R_1\cos\theta_1 & \ldots & \beta_o \\ 1 & 0 & R_2\cos\theta_2 & -R_2\sin\theta_2 & \ldots & \alpha_1 \\ 0 & 1 & R_2\sin\theta_2 & R_2\cos\theta_2 & \ldots & \beta_1 \\ & & \cdot & & & \\ & & \cdot & & & \\ & & \cdot & & & \\ 1 & 0 & R_{2k}\cos\theta_{2k} & -R_{2k}\sin\theta_{2k} & \ldots & \alpha_{2k} \\ 0 & 1 & R_{2k}\sin\theta_{2k} & R_{2k}\cos\theta_{2k} & \ldots & \beta_{2k} \end{bmatrix} \quad (18)$$

where from $2k$ nodal points, a $(k-1)$ order polynomial $\xi_f(z)$ will be developed for Ω_f. An analogous development applies for the complex polynomial $\xi_t(z)$.

The full system is written in matrix form

$$K(\mathbf{T}, \mathbf{Q}) = 0 \quad (19)$$

where $\mathbf{K}$ is a fully populated matrix of known coefficients from equation (18) and $(\mathbf{T}, \mathbf{Q})$ is the array of (T_i, Q_i) values of the complex temperature ξ_f. Heat-flux values can be estimated along Γ^* directly from equation (17b).

For an anisotropic homogeneous problem, the methodology in equation (19) can be utilized by rescaling the global problem to accommodate the ratio of horizontal-to-vertical thermal conductivity values (Myers [7]), and solving the modified problem in the new $(\hat{x}, \hat{y})$ space.

For homogeneous problems, complexities arise in an effort to simultaneously solve for the unknowns of ξ_i, shared on the boundaries of homogeneous regions. Several nodal points are required interior of Ω along the boundaries of the defined *homogeneous regions, resulting in a significant increase* in computational effort due to the fully populated matrix requirements of a numerical formulation. In this model, the method used for nodal point placement along the boundary contours is to evenly locate the nodal points and add additional nodes uniformly to the contours until the complex polynomial coefficients (α_i, β_i) begin to show negligible change. Comparison of modeled results to analytic solutions of several harmonic functions indicated that good estimates of flux values and unknown T or Q values are produced at the given boundary nodal points.

To show that $\{w(z):\ w(z) = \xi_f(z),\ \xi_t(z)\}$ satisfies the governing Laplace PDE follows from elementary complex variable theory of analytic functions (e.g., Churchill [2]). Consideration of singularities occurring within the radius of convergence of $w(z)$ can be addressed by expanding the $w(z)$ polynomial about another point interior of Ω such that the boundary of $\Omega = \Gamma_f U \Gamma_t$ lies entirely within the assumed radius of convergence of the true solution of the governing PDE.

Model Verification

The two-dimensional model was tested against a one-dimensional freezing column experiment given in Jame [6]. The soil used was a fine-grained Silica Flour, and detailed heat and soil-water flow parameter data are given in that study.

The model was applied to the one-dimensional test problem by using a six nodal point model shown in Fig. 3. A single finite strip is used to approximate soil-water flow in the unfrozen subregion. Separate Laplace approximations are used in each of the frozen and unfrozen subregions. The initial moisture content of the soil was 15 percent (dry weight) with boundary conditions as plotted in Figs. 4 and 5.

Difficulty was encountered in the initial portions of the simulation due to the relative rapid movement of the freezing front. To avoid this difficulty, the inital condition of the test was taken to be the experimental results for time at 6 hr. Although using very small timesteps (0.01 hr) reduced the approximation error, the computer results continued to overestimate the freezing front penetration by about 20 percent during the initial 6 hr of simulation.

Figures 4 and 5 compare the experimental results of Jame [6] for the freezing column and the model results using the six

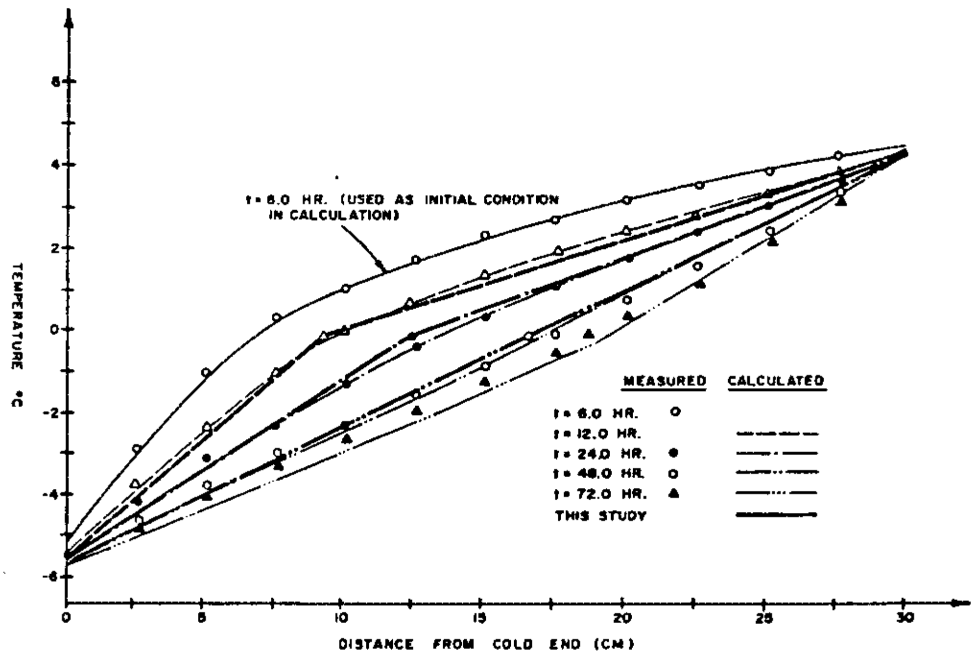

Fig. 4 Calculated temperature profiles compared with experimental results (Jame [6])

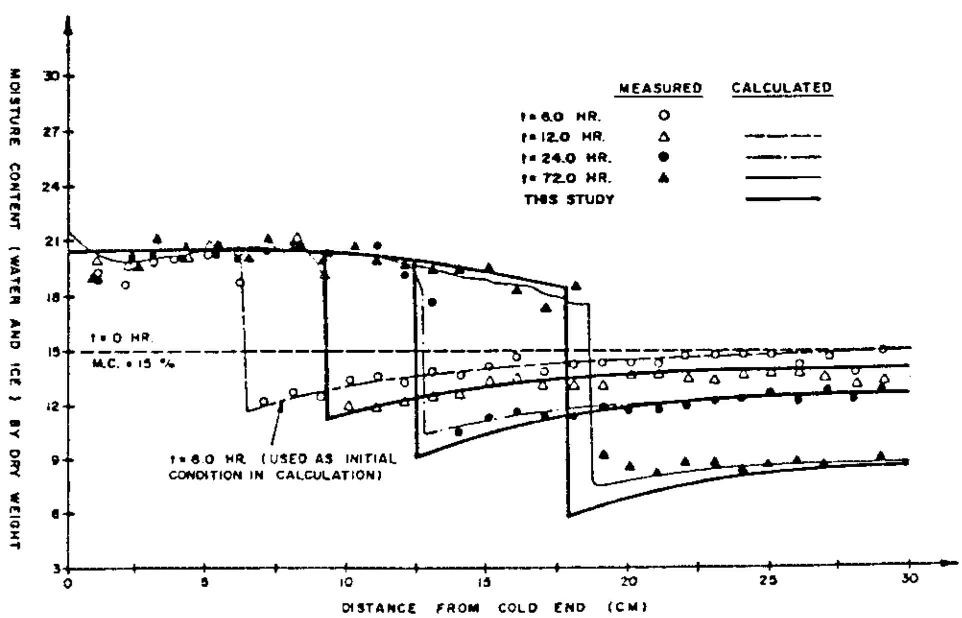

Fig. 5 Calculated moisture content profiles compared with experimental results (Jame [6])

Table 1 **Modeling results and laboratory frost-heave data**

Time (days)	Domain model[a] (cm)	Ice segregation model (cm)	Laboratory data (cm)
5	1.5	1.3	1.6
10	2.9	2.6	2.8
15	3.9	3.9	3.9
20	4.6	4.8	4.4
25	5.0	5.4	5.0
30	5.2	5.8	5.6
35	5.4	6.0	5.8
40	5.4	6.1	5.8
45	5.5	6.2	5.8
110	5.6	6.4	5.8

[a]Guymon and others [3]

nodal point scheme. In this test, no frost heave was predicted which agreed with the experimental results.

In order to obtain the given results, the soil-water conduction parameter had to be reduced to about 0.07 of its value as determined by the thawed unsaturated conduction parameter of equation (4). This parameter modification compares to the reduction values of 0.05 to 0.001 used by Jame [6] in his finite difference model based on the theory given in Harlan [4]. Other model hydraulic parameter modification formulas are given in Taylor and Luthin [10] and Guymon and others [3] which exponentially reduce the soil-water flow conduction parameter as a function of ice content.

From Fig. 5, the total moisture content begins to deviate from the experimental results as time continued. This discrepancy was significantly reduced by varying the required latent heat budget specified at the freezing front domain. The results of Fig. 5 are based on a constant coefficient of latent heat of 80 cal/cm^3. It can be noted that this test case essentially involved only a dewatering of a soil column and, consequently, is testing only the simple dewatering algorithm of the solid-water flow model.

To examine a freezing column problem where a water table is of concern, the domain model of Guymon and others [3] was tested against the model of Fig. 3. Using parameter information of a Fairbanks silt and identical boundary condition information, both models predicted values of freezing front penetration into the soil and frost-heave development. Both modeling results for frost-heave are given in Table 1. From Table 1, comparable results are produced by both models, although the computational effort is significantly reduced by the proposed model.

Conclusions

A simple numerical model to predict ice segregation by means of a coupled heat and soil-water flow analysis is developed. The model can be prepared for use with a significant reduction in coding over current domain type models. The model produces reasonable predictions of frost-heave development and the location of freezing fronts. The model predicts values of heat and soil-water flux directly at the freezing front without the regeneration of a global two-dimensional finite element on finite difference mesh.

The model is seen to be a strong function of the impedance factor used to reduce soil-water flow at the freezing front. Although models use such impedance factors to approximate soil-water flow effects in freezing soils, a definite procedure to estimate such a factor is not given in the literature. However, this factor may be considered a model calibration parameter which is determined by attempts to match one-dimensional column data. After calibration, the model can be used in approximating two-dimensional problems, where the two-dimensional domain is composed of the material studied by the one-dimensional column tests.

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