Complex Polynomial Approximation of the LaPlace Equation

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Abstract: A method of approximating the solution of the Laplace equation in two-dimensions is presented. The numerical approach is to determine a complex variable polynomial which satisfies the specified boundary conditions along a simple closed contour. Since the method is simple to apply to time-stepped, quasi-steady state saturated ground water problems or moving boundary problems, a significant savings in computational effort over other boundary integral equation methods is available. Applications to a free water surface problem and a moving boundary problem are presented. Error bounds and model stability are considered.

Introduction

The purpose of this paper is to develop an application of complex variable analytical function theory to the approximation of the two-dimensional Laplace equation

$$\frac{\partial^2 \xi(x,y)}{\partial x^2} + \frac{\partial^2 \xi(x,y)}{\partial y^2} = 0, \quad (x,y) \in \Omega \qquad (1)$$

in which $\xi(x,y)$ is a two-dimensional harmonic function defined in global domain Ω with global boundary Γ. This type of analysis differs from the usual domain numerical methods or boundary integral equation methods (B.I.E.M.) in that a complex approximation function naturally satisfies the governing partial differential equation (P.D.E.) and a weighted residual minimization is, therefore, not required. Dirichlet, Neuman, and mixed boundary conditions along the global boundary are used to develop the P.D.E. complex polynomial approximator. Interior nodal points are not required although inclusion of domain nodal points can be used to extend the concepts presented here.

B.I.E.M. models based on real variable theory are well known and have been applied to two and three-dimensional problems involving the solution to Dirichlet (specified boundary state variable) and Neumann (specified boundary state variable normal gradient) problems (6,5). Brebbia (1) generalizes B.I.E.M. modeling to a boundary element method which has been applied to harmonic function approximations. Liggett and Liu (8) apply B.I.E.M. models to time dependent free-surface flow in porous media.

Formulation

Recently, Hunt and Isaacs (4) utilize the Cauchy's integral theorem to develop a B.I.E.M. model for solution of the Laplace equation. For a

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complex function $\omega(z)$ analytic inside a simple closed contour C, Cauchy's theorem relates the value of $\omega(a)$ to the boundary integral

$$\omega(a) = \frac{1}{2\pi i} \oint_C \frac{\omega(z)}{(z-a)} dz, \quad i = \sqrt{-1} \quad \ldots \ldots \ldots \ldots \ldots (2)$$

where point (a) is interior of contour C, and contour C is integrated in the positive (counter-clockwise) direction. In Eq. 2, the analytical function $\omega(z)$ is composed of two harmonic real variable two-dimensional functions $\phi(x,y)$ and $\psi(x,y)$

$$\omega(z) = \phi(x,y) + i\psi(x,y) \quad \ldots \ldots \ldots \ldots \ldots (3)$$

in which $z = x + iy$, and $\phi(x,y)$ and $\psi(x,y)$ are orthogonal functions that satisfy the usual definitions of state variable and stream function, respectively; for example, $\phi(x,y)$ are the potential functions in a saturated porous media flow problem and $\psi(x,y)$ are the stream functions.

The harmonic functions associated to the complex function $\omega(z)$ are related by the Cauchy-Riemann relations (2)

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \ldots \ldots \ldots \ldots \ldots (4)$$

and $$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \quad \ldots \ldots \ldots \ldots \ldots (5)$$

Along an isopotential or streamline, normal flux or the potential gradient is calculated from Eqs. 4 and 5 in which tangential and normal distance is used rather than the (x,y) coordinates.

From the above considerations, Dirichlet, Neumann, or mixed problems can be studied using the Cauchy integral theorem to formulate a B.I.E.M. model. The procedure is to first subdivide the global boundary Γ into segments by nodal points. It is assumed that either values of ψ or ϕ are specified at each nodal point. In their model, Hunt and Isaacs (4) assume that $\omega(z)$ is linear between successive nodal points such that

$$\omega(z) = \left(\frac{z - z_j}{z_{j+1} - z_j}\right) \omega_{j+1} + \left(\frac{z_{j+1} - z}{z_{j+1} - z_j}\right) \omega_j \quad \ldots \ldots \ldots \ldots \ldots (6)$$

in which $\omega_j = \omega(z_j)$ and $z \epsilon [z_j, z_{j+1}]$ in which the contour integration is in the positive sense. A contour integration along global boundary Γ is made for each nodal point on Γ such that the Cauchy principal value results in a linear expression of the unknown nodal variable as an implicit function of all unknown nodal variables on Γ.

The contribution from all boundary nodal points results in a fully populated square matrix

$$\mathbf{K}(\boldsymbol{\phi}, \boldsymbol{\psi}) = \mathbf{0} \quad \ldots \ldots \ldots \ldots \ldots (7)$$

in which $\mathbf{K}$ is an $N \times N$ matrix for N nodal points and $(\boldsymbol{\phi}, \boldsymbol{\psi})$ is an array of unknown values at the boundary nodes. Mathematically, the above formulation can be expressed by using a complex variable approximator $\hat{\omega}(z)$ of the true solution $\omega(z)$ which is defined on Γ such that

$$\oint_{\Gamma} \frac{\omega(z)dz}{(z - z_0)} = \sum_j \oint_{\Gamma_j} \frac{\hat{\omega}(z)dz}{(z - z_0)} \quad (8)$$

in which Γ_j is a segment of Γ between two successive nodes on Γ.

An alternative approach is developed using a complex polynomial approximation on Γ instead of using a contour integration of Cauchy's theorem. The requirement of $\hat{\omega}(z)$ being linear between nodal points in Eq. 8 can be extended to $\omega(z)$ being a complex polynomial of order $(k - 1)$ where $2k$ nodal points are specified on Γ. In this case, it is assumed that $2k$ boundary conditions are known along Γ, such as either ψ or ϕ at specified nodal points.

For example, given a six nodal point discretization of Γ (Fig. 1) a second order complex polynomial approximation $\hat{\omega}_2(z)$ can be defined on Γ (and therefore in Ω)) such that

$$\hat{\omega}_2(z) = (\alpha_0 + i\beta_0) + (\alpha_1 + i\beta_1)z + (\alpha_2 + i\beta_2)z^2 \quad (9)$$

By Cauchy's integral theorem,

$$2\pi i\, \hat{\omega}_2(z_0) = \oint_{\Gamma} \frac{\hat{\omega}_2(z)dz}{z - z_0}, \quad z_0 \in \Omega \quad (10)$$

Combining Eqs. 8, 9 and 10,

$$2\pi i\, \hat{\omega}_2(z_0) = (\alpha_0 + i\beta_0) \oint_{\Gamma} \frac{dz}{(z - z_0)} + (\alpha_1 + i\beta_1) \oint_{\Gamma} \frac{zdz}{(z - z_0)}$$

$$+ (\alpha_2 + i\beta_2) \oint_{\Gamma} \frac{z^2dz}{(z - z_0)} \quad (11)$$

Eq. 11 can be expanded by partial fractions and simplified into the expression

$$2\pi i\, \hat{\omega}_2(z_0) = [(\alpha_0 + i\beta_0) + z_0(\alpha_1 + i\beta_1) + z_0^2(\alpha_2 + i\beta_2)] \oint_{\Gamma} \frac{dz}{(z - z_0)}$$

$$+ [(\alpha_1 + i\beta_1) + z_0(\alpha_2 + i\beta_2)] \oint_{\Gamma} dz + (\alpha_2 + i\beta_2) \oint_{\Gamma} zdz \quad (12)$$

For z_0 interior of Γ,

$$\oint_{\Gamma} \frac{dz}{(z - z_0)} = 2\pi i; \quad z_0 \in \Omega \quad (13)$$

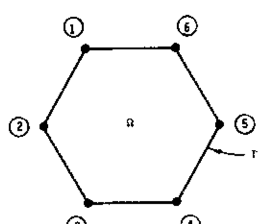

FIG. 1.—Six Point B.I.E.M. Discretization of Global Boundary Γ

And for any z_0, the complex functions z and z^2 are analytic in the entire complex plane and

$$\oint_\Gamma dz = \oint_\Gamma z dz = 0 \quad (14)$$

From Eqs. 12, 13, and 14,

$$\hat{\omega}_2(z_0) = (\alpha_0 + i\beta_0) + z_0(\alpha_1 + i\beta_1) + z_0^2(\alpha_2 + i\beta_2) \quad (15)$$

That is, a complex polynomial approximation on Γ can be determined by simply evaluating the complex polynomial on Γ rather than using the contour integration of Cauchy's theorem to evaluate the approximation coefficients. In Eq. 15, $\hat{\omega}_2(z_0)$ can be evaluated for $z_0 \epsilon \Gamma$ as well as $z_0 \epsilon \Omega$. Additionally, the approximator $\hat{\omega}_2(z)$ is based on a higher order polynomial interpolation on Γ rather than a linear interpolation between nodal points.

Using Eq. 15 to evaluate the unknown coefficients of an approximator function $\hat{\omega}(z)$ on Γ results in a straightforward method to determine the unknown values of ϕ or ψ on Γ. Using polar coordinates (Fig. 2), the Euler formula describes a complex point z by

$$z = x + iy = R(\cos\theta + i\sin\theta) = Re^{i\theta} \quad (16)$$

in which $R^2 = x^2 + y^2$, $\theta = \arctan(y/x)$. By de Moivre's theorem,

$$z^k = R^k e^{ik\theta}, \quad k = 0, 1, 2, \ldots \quad (17)$$

or in simpler terms,

$$z^k = R^k(\cos k\theta + i\sin k\theta) \quad (18)$$

From the previous, expansions of complex polynomial order k is made for $2(k + 1)$ nodal points by

$$\phi(z) = \alpha_0 + (\alpha_1 R\cos\theta - \beta_1 R\sin\theta)$$
$$+ (\alpha_2 R^2\cos 2\theta - \beta_2 R^2\sin 2\theta) + \ldots + (\alpha_k R^k\cos k\theta - \beta_k R^k\sin k\theta) \quad (19)$$

$$\psi(z) = \beta_0 + (\beta_1 R\cos\theta + \alpha_1 R\sin\theta)$$
$$+ (\beta_2 R^2\cos 2\theta + \alpha_2 R^2\sin 2\theta) + \ldots + (\beta_k R^k\cos k\theta + \alpha_k R^k\sin k\theta) \quad (20)$$

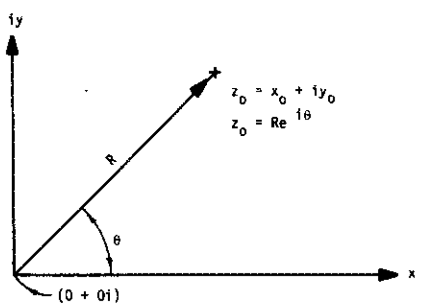

FIG. 2.—Polar Coordinate Definition of Complex Point z_0

In matrix form, Eqs. 19 and 20 are evaluated for each node to give

$$\begin{Bmatrix} \phi_1 \\ \psi_1 \\ \phi_2 \\ \psi_2 \\ \vdots \\ \phi_{2k+2} \\ \psi_{2k+2} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & R_1 \cos\theta_1 & -R_1 \sin\theta_1 & \cdots \\ 0 & 1 & R_1 \sin\theta_1 & R_1\cos\theta_1 & \cdots \\ 1 & 0 & R_2\cos\theta_2 & -R_2\sin\theta_2 & \cdots \\ 0 & 1 & R_2\sin\theta_2 & R_2\cos\theta_2 & \cdots \\ & & & \vdots & \\ 1 & 0 & R_{2k+2}\cos\theta_{2k+2} & -R_{2k+2}\sin\theta_{2k+2} & \cdots \\ 0 & 1 & R_{2k+2}\sin\theta_{2k+2} & R_{2k+2}\cos\theta_{2k+2} & \cdots \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \beta_0 \\ \alpha_1 \\ \beta_1 \\ \vdots \\ \alpha_k \\ \beta_k \end{Bmatrix} \quad (21)$$

in which in Eq. 21 only the known nodal values of either ϕ or ψ are used. Solving for the (α_i, β_i) gives a kth order complex polynomial which satisfies the Laplace equation in Ω and the given boundary conditions at specified locations on Γ.

A significant advantage of the simple formulation of Eq. 21 over other B.I.E.M. formulations is that each row of the global matrix does not require a complete circuit of the boundary integral to determine the matrix entries. This advantage reduces overall computational effort and allows for an improvement in computational efficiency for time dependent problems wherein the global matrices are frequently regenerated. Another advantage is that the proposed model is simple in application. Finally, the proposed method offers a higher order polynomial approximation $\hat{\omega}(z)$ of the true problem solution $\omega(z)$ with significantly less computational effort than a linear interpolation model.

The time savings obtained by use of the simpler global matrix construction of Eq. 21 becomes apparent when using the model to approximate state variable and normal flux values in a moving boundary problem (normal flux values can be computed by the Cauchy-Riemann equations modified with respect to normal and tangential coordinates). Many problems of interest include either an iterative scheme or a time-stepped quasi-steady state solution to a time-dependent P.D.E. which require several global matrix regenerations. Consequently, the simpler global matrix of Eq. 21 offers a method to reduce overall model costs for many problems.

Model Applications

Three problems will be presented in order to demonstrate the use of the proposed model. For comparison purposes, results from the model of Hunt and Isaacs (4) will also be given. Discussion of stability and convergence tendencies will be given in the following section.

Example Problem Number One: Approximation of Complex Exponential Function.—The first example is of a numerical approximation for the complex variable transcendental function $\omega = \exp(z)$. In this test problem, the global domain of definition Ω is defined as (Fig. 3)

$$\Omega:\{(x,y)|0 \le x \le 2, 0 \le y \le 2\} \quad (22)$$

The boundary conditions are specified along the global boundary, Γ, where a 16-nodal point distribution is used on Γ. The state variable $\phi(x,y)$ was specified along Γ except for a single nodal point where the stream function $\psi(x,y)$ was specified. Consequently, 15 values of $\psi(x,y)$ and a single

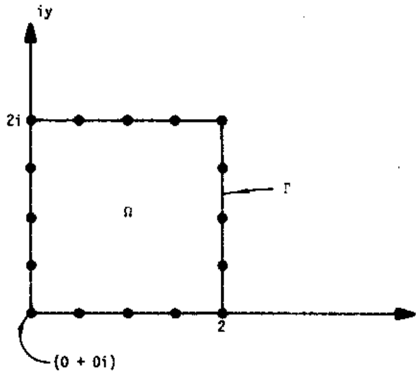

FIG. 3.—Geometry of Example Problem #1

value of $\phi(x,y)$ remain to be determined on Γ.

The functions $\phi(x,y)$ and $\psi(x,y)$ are given by

$$\phi(x,y) = e^x \cos y \quad (23)$$

$$\psi(x,y) = e^x \sin y \quad (24)$$

Comparison of model results to the solutions of Eqs. 23 and 24 indicate that the proposed polynomial model provides an improvement in approximation accuracy. Table 1 contains a comparison of solutions to this test problem.

Example Problem Number Two: Groundwater Phreatic Surface Approximation.—The estimation of the phreatic groundwater surface is of

TABLE 1.—Comparison of Modeled Solutions to Analytic Solution of Test Problem #1

Node number (1)	x (2)	y (3)	Unknown variable (4)	Cauchy model (5)	Polynomial model (6)	Analytic solution (7)
1	0	0	ψ	−0.20	0.00	0
2	0.5	0	ψ	−0.12	−0.00	0
3	1.0	0	ψ	−0.07	−0.00	0
4	1.5	0	ψ	0.02	0.00	0
5	2.0	0	ψ	0.57	−0.00	0
6	2.0	0.5	ψ	3.59	3.54	3.54
7	2.0	1.0	ψ	6.15	6.22	6.22
8	2.0	1.5	ψ	7.29	7.37	7.37
9	2.0	2.0	ψ	6.81	6.72	6.72
10	1.5	2.0	ψ	3.96	4.08	4.08
11	1.0	2.0	ψ	2.36	2.47	2.47
12	0.5	2.0	ψ	1.39	1.50	1.50
13	0	2.0	ψ	0.78	0.91	0.91
14	0	1.5	ψ	0.92	1.00	1.00
15	0	1.0	ψ	0.84	0.84	0.84
16	0	0.5	ϕ	0.88	0.88	0.88

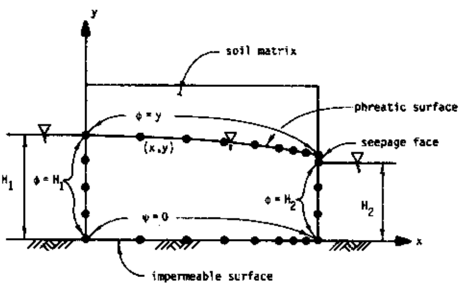

FIG. 4.—Definition of Phreatic Surface Groundwater Problem Showing Nodal Point Placement

interest in many problems in groundwater flow (7). In this example problem, a simple case of two levels of ponded water separated by a uniform homogeneous soil is considered in which the objective is to predict the location of the phreatic water surface. The problem definition and boundary conditions are shown in Fig. 4 (10).

The problem is complicated by the existance of a seepage surface such that the flowlines and potentials of the problem satisfy the Laplace equation. The algorithm used for both the Cauchy integral and the complex polynomial model is to define by trial and error values of y along the phreatic surface until the computed ϕ-values on the phreatic surface coincide with the y-coordinates at the specified nodal points.

The results of both complex variable theory analogs are compared to the steady state free water surface (phreatic) presented in Vauclin, et al. (10). Also compared are the domain model results for the steady state condition presented in Nasasimhan (9). From the several comparisons (Fig. 5), both complex variable theory models produced results of similar accuracy with respect to relative error in the prediction of the free water surface.

Due to the global matrix regeneration requirements of the algorithm, the complex polynomial model was found to reduce overall computa-

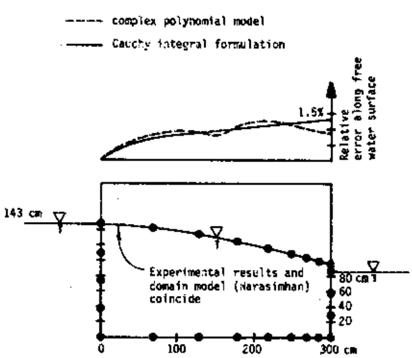

FIG. 5.—Steady State Free Water Location Using Various Methods and Complex Variable Model Error

tional effort by one-third of that required for the Cauchy integral model to converge to the algorithm specifications. Such savings in computational effort becomes attractive when studying time dependent problems such as in domain interface studies. Some classes of these problems include salt-water intrusion into ground water aquifers and moving boundary soil water freezing problems.

Example Problem Number Three: Moving Boundary Problem.—The moving boundary problem in naturally freezing soils has been studied by many investigators. Hromadka and Guymon (3) developed a geothermal model based on the Cauchy integral model of Hunt and Isaacs (4). Due to an extremely slow moving freezing front, the classical heat equation was simplified to a Laplace equation to be solved by the Cauchy integral method. In their model, application of a B.I.E.M. model was found to be advantageous over a domain model due to a moving freezing front which is modeled as an isotherm. Consequently, the Cauchy-Riemann equations coupled with the B.I.E.M. model gave values of heat flux along the freezing front which are then used to relocate the freezing front according to a simple temporal integrated balance of volumetric soil-water phase change and net heat flux evolution.

In this example, the Cauchy integral model and the complex polynomial model are compared in computational efficiency in solving the moving boundary freezing front problem. Since results were very similar, the main objective was to examine overall computational cost. The problem domain and boundary conditions are shown in Fig. 6.

The Laplace equation is solved in the frozen area assuming no heat flux below the freezing front. After computing the heat flux along the freezing front, the change in coordinates is computed based on the time-

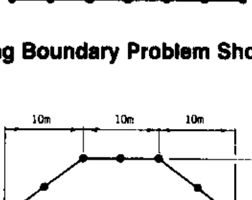

FIG. 6.—Definition of Moving Boundary Problem Showing Nodal Point Placement

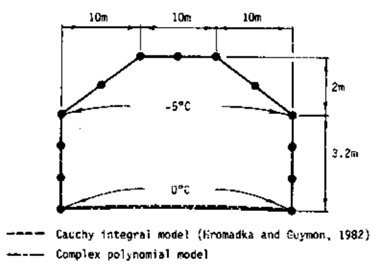

FIG. 7.—10 Year Simulation of Freezing Front Location for Soil Freezing Problem

step size of one-day. Then, a new global matrix is developed based on the new solution domain geometry.

The predicted freezing front locations from the complex polynomial model differed less than 2% from results obtained from the Cauchy integral model. However, for a 10 yr simulation (Fig. 7) the complex polynomial model used 35% less computation costs than the Cauchy integral model used in Hromadka and Guymon (3).

Based on the example problems, a strong case can be made for the use of the proposed complex polynomial model. A further sophistication of the model is to include interior points in the polynomial determination or weighted residual methods. However, further research is required in order to better qualify these sophistications.

Stability and Convergence Considerations

Use of the proposed complex polynomial approximation method (C.P.A.M.) for the solution of Dirichlet and Neumann problems requires some insight into the stability and convergence capabilities of the model. First, the problem considered must be based on a Laplace equation with no singularities on Γ or in Ω. Should singularities of the solution exist in Ω, then the $\hat{\omega}(z)$ approximation must have a pole or an essential singularity in order to ultimately converge to the required solution, $\omega(z)$. Since the current model assumes that $\omega(z)$ is analytic on Γ and in Ω, there can be no singularities and

$$\lim_{n\to\infty} \hat{\omega}_n(z) = \omega(z) \quad \text{(25)}$$

in which n is the complex polynomial order, and it is assumed that the nodal point distribution on Γ is uniform such that the contour length between successive nodal points is a constant for any order, n.

For the model based on the Cauchy integral theorem, the convergence also follows from Eq. 25 when the nodal point distribution is again uniform on Γ. In the C.P.A.M. model, the user must determine an appropriate order N such that

$$|\hat{\omega}_n(z) - \hat{\omega}_{n+1}(z)| < \epsilon; \quad n \geq N,\ z \epsilon \Gamma \quad \text{(26)}$$

in which ϵ is a tolerance, and it is assumed that this tolerance is applicable on the domain, Ω. In Eq. 26, Cauchy's convergence criteria for converging power series is used to determine N.

The error of approximation in domain Ω can be estimated by noting from Eq. 26

$$|\hat{\omega}_N(z) - \omega(z)| < \epsilon, \quad z \epsilon \Gamma \quad \text{(27)}$$

Therefore, for some complex value $(a)\ \epsilon \Omega$,

$$\hat{\omega}_N(a) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{\hat{\omega}_N(z)\,dz}{(z-a)} \quad \text{(28)}$$

Consequently, the error at point (a) is given by

$$\omega(a) - \hat{\omega}_N(a) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{\omega(z)\,dz}{(z-a)} - \frac{1}{2\pi i} \oint_{\Gamma} \frac{\hat{\omega}_N(z)\,dz}{(z-a)} \quad \text{(29)}$$

or in a simpler formulation,

$$|\omega(a) - \hat{\omega}_N(a)| \leq \frac{1}{2\pi} \oint_\Gamma \frac{|\omega(z) - \hat{\omega}_N(z)||dz|}{|(z - a)|} \quad (30)$$

From Eqs. 27 and 30,

$$|\omega(a) - \hat{\omega}_N(a)| \leq \frac{\epsilon L}{2\pi R} \quad (31)$$

where L = arc length of the contour Γ; and R is the smallest distance from point (a) to contour Γ. It should be noted that Eq. 31 does not imply that huge errors must occur close to the boundary, Γ, but that the error in the interior of Γ is bounded by the order N for the complex polynomial used. Obviously, since $\omega(z)$ and by necessity $\hat{\omega}_N(z)$ are both analytic on Γ and in Ω, then $\xi(z) = \omega(z) - \hat{\omega}_N(z)$ is analytic on Γ and in Ω and must therefore have its maximum modulus $|\xi(z)|$ on Γ (for a nonconstant $\omega(z)$). Therefore,

$$|\omega(a) - \omega_N(a)| < \epsilon, \ (a)\epsilon\Omega U\Gamma \quad (32)$$

One does not always know whether a problem includes a singularity and, consequently, whether $\omega(z)$ will converge. Several nodal densities may be requierd on Γ in order to ascertain whether the Cauchy convergence criteria is satisfied. Depending on computer roundoff error problems, a high order approximation (e.g., $N = 40$) may be unobtainable and yet the approximation error is unacceptable. In such cases, the C.P.A.M. model fails and other techniques should be used.

The writers have found that the C.P.A.M. is sensitive to nodal point placement and that lower order C.P.A.M. models can be developed which meets the acceptable accuracy requirements of Eq. 26 depending on certain concentrations of nodes on Γ. In contrast, instability can occur for irregular nodal placement on Γ for even high polynomial orders.

Conclusions

A new numerical model of the well known Laplace equation is advanced. The method is based on a complex variable polynomial expansion which satisfies the specified boundary conditions. The method is related to B.I.E.M. techniques, but offers a significant reduction in computational effort.

An attractive application of the method is to moving boundary problems such as in two-dimensional soil-water freezing or thawing. In this class of problems, the governing time-dependent P.D.E. is simplified to a time-stepped quasi-steady state problem. The solution of the simplified P.D.E. provides state variable normal flux values along the moving boundary which can be used to calculate the corresponding boundary displacement. This simple approach can be extended to other two-dimensional moving boundary or interface problems such as salt-water ground water intrusion studies or problems of coupled heat and soil water flow in freezing soils.

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