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TECHNICAL NOTE

LINKING THE COMPLEX VARIABLE BOUNDARY-ELEMENT METHOD TO THE ANALYTIC FUNCTION METHOD

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The complex variable boundary-element method (CVBEM) is expanded into a finite series of analytic functions. For the special case of linear polynomial basis functions assumed on each boundary element, the CVBEM and the analytic function method (AFM) results in a similar numerical modeling approximation.

INTRODUCTION

Several advances in the use of boundary integral equation methods to solve boundary-value problems such as the Laplace equation Dirichlet and Neumann problems have been reported. The major thrust in this modeling technique has been in real variable methods based on Green's theorems (e.g., [1]).

By using complex variable analytic function theory, however, exact solutions of the boundary-value problem are achieved interior to the approximate boundary [2]. Use of analytic function theory including potential function analysis for the study of two-dimensional steady-state heat transfer problems is well advanced [3]. The analytic function method (AFM) [2] is an approach to developing an analytic function approximation of the true solution to the boundary-value problem. The complex variable boundary-element method (CVBEM) [4] develops another analytic function approximation by using a Cauchy integral to develop a boundary integral equation.

In this paper, the CVBEM will be shown to expand into a series of products of complex polynomial and logarithmic functions. The order of these complex polynomials will be shown to equal the order of the specified polynomial basis functions. The logarithmic functions will be shown to be centered about each nodal point specified on the problem boundary. Finally, the CVBEM will be shown to reduce to the AFM for the special case of linear basis functions assumed between boundary nodal points. This final result provides a direct and valuable link between the two numerical modeling approaches.

Because this note provides an analogy between the CVBEM and the AFM, the theoretical foundations developed for the AFM (such as applications for inhomogeneous and anisotropic materials) can be immediately applied to the more recently developed CVBEM. In addition, the capability of the CVBEM to utilize higher order trial functions can now be incorporated into the AFM by using the derived linking formulas. Consequently, computer programs based on either method can be modified to represent either modeling approach and to incorporate the benefits of both numerical methods.

NOMENCLATURE

$G(z)$	global trial function	z_j	nodal coordinate
$N_{j,i}^k(s)$	basis function	$\bar{\omega}_{j,i}$	nodal value
s	local coordinate	$\omega(z)$	analytic solution
z	complex coordinate	$\hat{\omega}(z)$	approximate function

MATHEMATICAL DEVELOPMENT

Let Ω be a simply connected domain with boundary Γ, where Γ is a simple closed contour. Discretize Γ by m nodal points into m boundary elements such that a node is placed at every angle point on Γ. Each boundary element is defined by

$$\Gamma_j = \{z : z = z(s) \text{ where } z(s) = z_j + (z_{j+1} - z_j)s \quad 0 \leqslant s \leqslant 1\} \quad j \neq m \tag{1}$$

with the qualification that

$$\Gamma_m = \{z : z = z(s) \text{ where } z(s) = z_m + (z_1 - z_m)s \quad 0 \leqslant s \leqslant 1\}$$

Then

$$\Gamma = \bigcup_{j=1}^{m} \Gamma_j \tag{2}$$

In the following, the qualification for element m [such as in the definition of Eq. (1)] will be omitted whenever it is understood. In addition, when understood, the limits of summation will not be stated, so that

$$\Gamma = \bigcup_{j=1}^{m} \Gamma_j \equiv \bigcup_{j} \Gamma_j \tag{3}$$

Let each Γ_j be discretized by $(k + 1)$ evenly spaced nodes $(k \geqslant 1)$ such that Γ_j is subdivided into k equilength segments. Then Γ_j is said to be a $(k + 1)$-node element. Each Γ_j has an associated nodal coordinate system such that $z_{j,1} = z_j$ and $z_{j,k+1} = z_{j+1} = z_{j+1,1}$.

On each Γ_j, define a local coordinate system by

$$\zeta_j(s) = z_{j,1} + (z_{j,k+1} - z_{j,1})s = z_j + (z_{j+1} - z_j)s \quad 0 \leqslant s \leqslant 1 \tag{4}$$

where $d\zeta_j = (z_{j,k+1} - z_{j,1})\, ds$.

On each $(k + 1)$-node element Γ_j, a set of order k polynomial basis functions are defined by

$$N_{j,i}^k(s) = a_{j,i,0} + a_{j,i,1}s + \cdots + a_{j,i,k}s^k \tag{5}$$

where $i = 1, 2, \ldots, (k + 1)$ and $0 \leqslant s \leqslant 1$, and where

$$N_{j,i}^k\left(\frac{z_{j,n} - z_{j,1}}{z_{j,k+1} - z_{j,1}}\right) = \begin{cases} 1 & n = i \\ 0 & n \neq i \end{cases} \tag{6}$$

The basis functions are further defined to have the property that for $\zeta \in \Gamma$

$$N_{j,i}^{k}\left(\frac{\zeta - z_{j,1}}{z_{j,k+1} - z_{j,1}}\right) = \begin{cases} N_{j,i}^{k}\left(\frac{\zeta - z_{j,1}}{z_{j,k+1} - z_{j,1}}\right) & \zeta \in \Gamma_j \\ 0 & \zeta \notin \Gamma_j \end{cases} \tag{7}$$

Let $\omega(z)$ be analytic on $\Omega \cup \Gamma$.

At each nodal point on Γ, define a specified nodal value by

$$\bar{\omega}_{j,i} = \omega(z_{j,i}) \tag{8}$$

Using Eqs. (7) and (8), an order k global trial function is defined by

$$G^k(\zeta) = \sum_j G^k[\zeta_j(s)] = \sum_j \sum_{i=1}^{k+1} \bar{\omega}_{j,i} N_{j,i}^k \ \frac{\zeta - z_j}{z_{j+1} - z_j} \tag{9}$$

From Eq. (9), the global trial function is continuous on Γ. An H_k approximation function $\hat{\omega}_k(z)$ is defined by the Cauchy integral

$$\hat{\omega}_k(z) = \frac{1}{2\pi i}\int_\Gamma \frac{G^k(\zeta)\,d\zeta}{\zeta - z} \qquad z \in \Omega \qquad z \notin \Gamma \tag{10}$$

Because the derivative of $\hat{\omega}_k(z)$ exists for all $z \in \Omega$, $\hat{\omega}_k(z)$ is analytic in Ω and exactly solves the Laplace equation in Ω.

The CVBEM continues from Eq. (10) by expanding

$$\int_\Gamma \frac{G^k(\zeta)\,d\zeta}{\zeta - z} = \sum_j \int_{\Gamma_j} \frac{G^k(\zeta)\,d\zeta}{\zeta - z} \tag{11}$$

But from Eq. (4)

$$\int_{\Gamma_j} \frac{G^k(\zeta)\,d\zeta}{\zeta - z} = \int_{s=0}^{1} \frac{\left[\sum_{i=1}^{k+1} \bar{\omega}_{j,i} N_{j,i}^k(s)\right](z_{j+1} - z_j)\,ds}{[z_j + (z_{j+1} - z_j)s] - z} \tag{12}$$

Rearranging Eq. (12),

$$\int_{\Gamma_j} \frac{G^k(\zeta)\,d\zeta}{\zeta - s} = \int_0^1 \frac{\sum_{i=1}^{k+1} \bar{\omega}_{j,i} N_{j,i}^k(s)\,ds}{s - \gamma_j} \tag{13}$$

where $\gamma_j = (z - z_j)/(z_{j+1} - z_j)$ is independent of the integration variable s.

Consider a quadratic set of basis functions. The integral of Eq. (13) is determined by noting from Eq. (5)

$$\int_0^1 \frac{\bar{\omega}_{j,i} N_{j,i}^2(s)\, ds}{s-\gamma_j} = \int_0^1 \frac{\bar{\omega}_{j,i}(a_{j,i,0} + a_{j,i,1}s + a_{j,i,2}s^2)\, ds}{s-\gamma_j}$$

$$= \bar{\omega}_{j,i}\left(\frac{a_{j,i,2}}{2} + a_{j,i,1} + a_{j,i,2}\gamma_j\right) + \bar{\omega}_{j,i}\bar{N}_{j,i}^2(\gamma_j)\ln\left(\frac{\gamma_j - 1}{\gamma_j}\right) \tag{14}$$

where $\bar{N}_{j,i}^2(\gamma_j)$ is the entire complex polynomial function determined by directly substituting γ_j into the variable s of $N_{j,i}^2(s)$, and

$$\ln\left(\frac{\gamma_j - 1}{\gamma_j}\right) = \ln\left(\frac{z - z_{j+1}}{z - z_j}\right) \tag{15}$$

From Eq. (14), the first set of resulting terms is an order 1 complex polynomial and the corresponding total integral of Eq. (13) will result in

$$\int_0^1 \frac{\sum_i \bar{\omega}_{j,i} N_{j,i}^2(s)\, ds}{s-\gamma_j} = R_j^1(z) + \sum_{i=1}^{k+1} \bar{\omega}_{j,i}\bar{N}_{j,i}^2(\gamma_j)\ln\left(\frac{z - z_{j+1}}{z - z_j}\right) \tag{16}$$

where $R_j^1(z)$ is an order 1 complex polynomial formed by the sum

$$R_j^1(z) = \sum_{i=1}^{k+1} \bar{\omega}_{j,i}\left(\frac{a_{j,i,2}}{2} + a_{j,i,1} + a_{j,i,2}\gamma_j\right) \tag{17}$$

The results of Eq. (16) can be directly extended to an order k approximation function by

$$\int_{\Gamma_j} \frac{G^k(\zeta)\, d\zeta}{\zeta - z} = R_j^{k-1}(z) + \sum_{i=1}^{k+1} \bar{\omega}_{j,i}\bar{N}_{j,i}^k(\gamma_j)\ln\left(\frac{z - z_{j+1}}{z - z_j}\right) \tag{18}$$

where $R_j^{k-1}(z)$ is an order $(k-1)$ complex polynomial.

Thus, the CVBEM results in an order k approximator $\hat{\omega}_k(z)$ which is defined by

$$\hat{\omega}_k(z) = \frac{1}{2\pi i}\sum_j \left[R_j^{k-1}(z) + \sum_{i=1}^{k+1} \bar{\omega}_{j,i}\bar{N}_{j,i}^k(\gamma_j)\ln\left(\frac{z - z_{j+1}}{z - z_j}\right)\right] \tag{19}$$

The objective is to expand Eq. (19) into a series of logarithmic functions. It is noted that the logarithmic contributions are of the form $\ln(z - z_j)$, $j = 1, 2, \ldots, m$, which involve only the endpoints of the boundary elements. In addition, each term $\ln(z - z_j)$ is associated with the two elements sharing z_j; namely, Γ_{j-1} and Γ_j. Rearranging Eq. (19),

$$\hat{\omega}_k(z) = \frac{1}{2\pi i}\sum_j \left[R_j^{k-1}(z) + \sum_{i=1}^{k+1} \bar{\omega}_{j,i}\bar{N}_{j,i}^k(\gamma_j)\ln(z-z_{j+1}) - \sum_{i=1}^{k} \bar{\omega}_{j,i}\bar{N}_{j,i}^k(\gamma_j)\ln(z-z_j)\right] \tag{20}$$

Combining terms according to $\ln(z-z_j)$ gives

$$2\pi i\hat{\omega}_k(z) = \left[\sum_j R_j^{k-1}(z)\right] + \left\{\sum_j \ln(z-z_j)\sum_{i=1}^{k+1}\left[\bar{\omega}_{j-1,i}\bar{N}_{j-1,i}^k(\gamma_{j-1}) - \bar{\omega}_{j,i}\bar{N}_{j,i}^k(\gamma_j)\right]\right\} + \left[2\pi i\sum_{i=1}^{k+1}\bar{\omega}_{m,i}\bar{N}_{m,i}^k(\gamma_m)\right] \tag{21}$$

In Eq. (21), the first term is the summation of order $(k-1)$ complex polynomials associated with the integration of Eq. (18). The second term is the logarithmic expansion associated with $\hat{\omega}_k(z)$. The third term accounts for the multiple-valued nature of the function $\ln(z-\zeta)$, where $\zeta \in \Gamma$ and the branch cut is assumed to originate from point $z \in \Omega$ and go through point $z_1 \in \Gamma$. Noting that the third term is an order k entire complex polynomial function, Eq. (21) can be rewritten as

$$\hat{\omega}_k(z) = R^k(z) + \frac{1}{2\pi i}\sum_j \ln(z-z_j)\sum_i T_{ij}^k \tag{22}$$

where $T_{ij}^k = \bar{\omega}_{j-1,i}\bar{N}_{j-1,i}^k(\gamma_{j-1}) - \bar{\omega}_{j,i}\bar{N}_{j,i}^k(\gamma_j)$, and $R^k(z)$ follows from Eq. (21).

By the definition of the polynomial basis functions in Eq. (7), $(z-z_j)$ is a factor of each $\bar{N}_{j-1,i}^k(\gamma_{j-1})$ except for $i=(k+1)$. Likewise, $(z-z_j)$ is a factor of each $\bar{N}_{j,i}^k(\gamma_j)$ except for $i=1$. For these two exceptions, $(z-z_j)$ is a factor of $[\bar{N}_{j-1,k+1}^k(\gamma_{j-1}) - \bar{N}_{j,1}^k(\gamma_j)]$. Thus, $(z-z_j)$ is a factor of each complex polynomial T_{ij}^k. Therefore, the final expansion form of $\hat{\omega}_k(z)$ is given by

$$\hat{\omega}_k(z) = R^k(z) + \sum_j P_j^{k-1}(z)(z-z_j)\ln(z-z_j) \tag{23}$$

where the $2\pi i$ term is combined into the general form of the order k complex polynomial $R^k(z)$.

From Eqs. (22) and (23),

$$P_j^{k-1}(z) = \frac{\sum_i T_{ij}^k}{z-z_j} \tag{24}$$

Two important characteristics of Eq. (23) that are utilized by the CVBEM are:

1. If $\omega(z)$ is an order k (or less) polynomial, then necessarily

$$\sum_i \bar{\omega}_{j,i} \bar{N}_{j,i}^k(\gamma_j) = \omega(z)$$

and from the definition of T_{ij}^k,

$$\sum_i T_{ij}^k = 0$$

Therefore, $\hat{\omega}_k(z) = \omega(z)$ implies $R^k(z) = \omega(z)$.

2. The limiting value for node $z_n \in \Gamma$ exists from

$$\lim_{z \to \bar{z}_n} \hat{\omega}_k(z) = R^k(z_n) + \sum_{\substack{j \\ j \neq n}} P_j^{k-1}(z_n)(z_n - z_j) \ln (z_n - z_j) \qquad (25)$$

This limiting value is used by the CVBEM to generate the necessary boundary integral equations for each nodal value [4].

The finite series of Eq. (25) can be used directly to model boundary-value problems of the Laplace equation by solving for the several complex polynomial coefficients given specified nodal point values at $z_j \in \Gamma$. The CVBEM of Eq. (10), however, models the boundary-value problem by solving directly for the potential or stream function nodal values. In Eq. (25), for the case $k = 1$, a form of the AFM [2] results.

CONCLUSION

The CVBEM results in a finite series of products of logarithmic functions and complex polynomials. The order of these polynomials equals the order of the assumed polynomial basis functions defined on each boundary element. For the special case of linear polynomial basis functions assumed on each boundary element, the CVBEM reduces to a form of the AFM.

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