

Determining relative error bounds for the CVBEM

T. V. HROMADKA II

US Geological Survey, Laguna Niguel, California, USA

The Complex Variable Boundary Element Method provides a measure of relative error which can be utilized to subsequently reduce the error or provide information for further modeling analysis. By maximizing the relative error norm on each boundary element, a bound on the total relative error for each boundary element can be evaluated. This bound can be utilized to test CVBEM convergence, to analyze the effects of additional boundary nodal points in reducing the modeling error, and to evaluate the sensitivity of resulting modeling error within a boundary element from the error produced in another boundary element as a function of geometric distance.

INTRODUCTION

The Complex Variable Boundary Element Method¹ utilizes analytic function theory to generate boundary integral equations as linear combinations of real coefficients multiplied by the boundary value problem's known and unknown nodal point boundary condition values. A definite advantage of the CVBEM to real variable based boundary element methods (BEM) or boundary integral equation methods (BIEM) is that the CVBEM results in linear equations which are exactly integrable, and a measure of relative error is generated by the CVBEM model. This relative error generation can then be utilized for modeling error analysis and model correction.

The main objectives of this paper are threefold:

- (1) To develop bounds on the relative error norm which include the relative error contributions from the unknown boundary condition values.
- (2) To develop a simplified relative error bound relationship which can subsequently be used to determine the impacts of error produced between boundary elements (as a function of relative distance between the elements), and
- (3) To evaluate convergence of the CVBEM and the response of the method to addition of nodal points on the problem boundary.

CVBEM DEVELOPMENT

Consider a simply connected domain Ω with simple closed contour boundary Γ . A boundary element discretization is generated by approximating Γ with m straight line segments (or boundary elements) Γ_j such that:

$$\Gamma \rightarrow \sum_{j=1}^m \Gamma_j \quad (1)$$

where in (1) Γ is not necessarily equal to the union, and $\Gamma_{j-1} \cap \Gamma_j = z_j = x_j + iy_j$ (Fig. 1). Each element can be

specified to have two (or more) nodal points such that a linear trial function (or higher order polynomial) will result in continuity of the state variable along Γ .

In this paper only the Laplace equation is studied with boundary conditions which are continuous along Γ such that there are no singularities on $\Omega \cup \Gamma$ and the solution to the boundary value problem, $\omega(z)$, is analytic on $\Omega \cup \Gamma$.

An analytic function $\omega(z)$ defined on $\Omega \cup \Gamma$ is composed of two harmonic real variable functions $\phi(x, y)$ and $\psi(x, y)$ such that:

$$\omega(z) = \phi(x, y) + i\psi(x, y) \quad (2)$$

where $\phi(x, y)$ and $\psi(x, y)$ satisfy the Cauchy-Riemann relations and

$$\nabla^2 \phi(x, y) = \nabla^2 \psi(x, y) = 0 \quad (3)$$

If $\omega(z)$ satisfies the problem boundary conditions on Γ , then $\phi(x, y)$ and $\psi(x, y)$ are the corresponding state and stream functions associated to the unique problem solution. For discussion purposes, the boundary value problem is

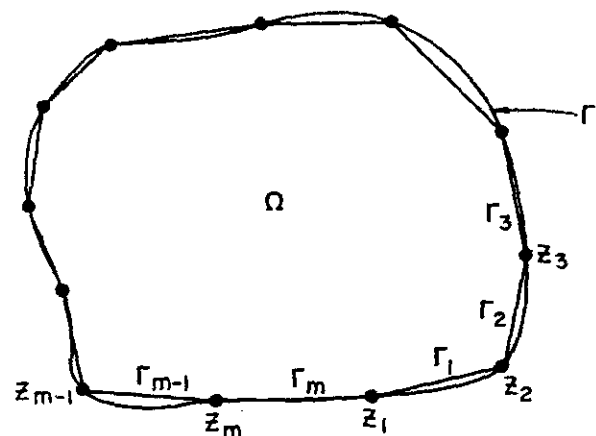


Figure 1. CVBEM boundary discretization

Accepted April 1984. Discussion closes August 1985.

assumed to have $\phi(x, y)$ specified continuously along Γ and a single point value of $\psi(x, y)$ is assumed specified.

Using the $\cup \Gamma_j$ approximation of Γ , nodal points can be specified at the ends of each element Γ_j such that:

$$\Gamma_j = \{z: z = z_j(1-s) + sz_{j+1}; 0 \leq s \leq 1\} \quad (4)$$

where (z_j, z_{j+1}) are the nodal point endpoint co-ordinates of Γ_j . At each nodal point j , a complex nodal value $\bar{\omega}_j$ can be specified by:

$$\bar{\omega}_j = \bar{\phi}_j + i\bar{\psi}_j \quad (5)$$

where $(\bar{\phi}_j, \bar{\psi}_j)$ are specified real numbers to be eventually associated to the real numbers (ϕ_j, ψ_j) where the following notation is used:

$$\omega(z_j) = \phi(x_j, y_j) + i\psi(x_j, y_j) = \phi(z_j) + i\psi(z_j) = \phi_j + i\psi_j \quad (6)$$

In (5) it should be noted that in our study case the $\bar{\phi}$ values are known for each node j , but the $\bar{\psi}$ values are unknown and need to be determined.

On each element Γ_j , define a trial function $\alpha_\phi + i\alpha_\psi$ such that:

$$\left. \begin{aligned} \alpha_\phi &= \alpha_\phi(\bar{\phi}, z) \\ \alpha_\psi &= \alpha_\psi(\bar{\psi}, z) \end{aligned} \right\} z \in \Gamma_j \quad (7)$$

In (7), α_ϕ (and α_ψ) are assumed to be continuous functions on Γ_j such that the function values between neighboring elements agree at the nodal points. In this construction, the trial functions for the unknown ψ functions are assumed to be simple linear polynomials:

$$\alpha_\psi = \bar{\psi}_j \left(\frac{z_{j+1} - z}{z_{j+1} - z_j} \right) + \bar{\psi}_{j+1} \left(\frac{z - z_j}{z_{j+1} - z_j} \right) \quad (8)$$

where in (8) α_ψ is used to approximate $\psi(z)$ in element j with $z \in \Gamma_j$.

An approximation function $\hat{\omega}(z)$ is developed by the integral function:

$$2\pi i \hat{\omega}(z) = \int_{\cup \Gamma_j} \frac{[\alpha_\phi + i\alpha_\psi] d\xi}{\xi - z}, \quad z \notin \Gamma \quad (9)$$

In (9) the integrand term $[\alpha_\phi + i\alpha_\psi]$ represents the various trial functions on each Γ_j as ξ is circuted along $\cup \Gamma_j$. For discussion purposes, let:

$$\Gamma = \bigcup_{j=1}^m \Gamma_j$$

where Γ is geometrically simple enough to be split into m straight line segments (or elements). Then:

$$2\pi i \hat{\omega}(z) = \int_{\Gamma} \frac{[\alpha_\phi + i\alpha_\psi] d\xi}{\xi - z} = \sum_{j=1}^m \int_{\Gamma_j} \frac{[\alpha_\phi + i\alpha_\psi] d\xi}{\xi - z} \quad (10)$$

where on each Γ_j , $[\alpha_\phi + i\alpha_\psi] = [\alpha_\phi(\bar{\phi}, z) + i\alpha_\psi(\bar{\psi}, z)]$.

Hromadka and Guymon² show that the principal value of (10) exists for $z \in \Gamma$ and

$$2\pi i \hat{\omega}(z_j) = \lim_{z \rightarrow z_j} \int_{\Gamma} \frac{[\alpha_\phi + i\alpha_\psi] d\xi}{\xi - z}, \quad j = 1, 2, \dots, m \quad (11)$$

where it is assumed m nodal points are specified on Γ . Consequently from (11), a linear equation is determined for each nodal point which relates the approximation function to a sum of specified nodal point values $\bar{\omega}$ multiplied by

complex variable coefficients. This system of equations can be written in matrix notation as:

$$\bar{\omega} = C_R(\bar{\phi}, \bar{\psi}) + iC_I(\bar{\phi}, \bar{\psi}) \quad (12)$$

where $\bar{\omega}$ is a vector of approximation function nodal values $\bar{\omega}(z_j) = \bar{\omega}_j = \bar{\phi}_j + i\bar{\psi}_j$. C_R and C_I are $m \times 2m$ matrices of real numbers representing the real and imaginary portions of the boundary integral equations, respectively.

In our problem, $\bar{\phi}$ is assumed known and $\bar{\psi}$ values are unknown (except for a single value to determine the constant of integration). Consequently, another rule or assumption is needed to determine the $2m$ unknown values of $\bar{\omega}$ given only m known values of $\bar{\omega} = \bar{\phi} + i\bar{\psi}$.

The rule used by Hunt and Isaacs³ and Hromadka and Guymon⁴ is to use only one of the matrix systems and force (in our case) $\hat{\psi} = \bar{\psi}$ by implicitly solving:

$$\hat{\psi} \equiv \bar{\psi} = C_I(\bar{\phi}, \bar{\psi}) \quad (13)$$

The result of this modeling approach is that:

$$\hat{\omega} = \bar{\phi} + i\hat{\psi} = \bar{\phi} + i\bar{\psi} \quad (14)$$

where in (14) the nodal point approximation values of the stream function $\hat{\psi}_j$ equal the now assumed nodal point boundary condition value, ψ_j . However, generally $\hat{\phi}_j \neq \bar{\phi}_j$. Thus, a relative error $e(z)$ is determined on Γ^- (where Γ^- is arbitrarily close but interior of Γ) by

$$\left. \begin{aligned} e(z) &= \omega(z) - \hat{\omega}(z), \quad z \in \Gamma^- \\ &= (\phi - \hat{\phi}) + i(\psi - \hat{\psi}) \end{aligned} \right\} \quad (15)$$

Hromadka and Guymon² present methods of working with this relative error in order to provide an overall reduction in modeling integrated error. In this paper, the main objectives are to develop estimated relative error bounds from this error function, and to develop simple ancillary integral equations which can be used to examine the error development in problem applications.

RELATIVE ERROR BOUND ESTIMATION

The relative error function $e(z)$ can be evaluated for some point $z_1^- \in \Gamma^-$ by:

$$2\pi i e(z_1^-) = \lim_{z \rightarrow z_1^-} \int_{\Gamma} \frac{[\omega(\xi) - (\alpha_\phi + i\alpha_\psi)] d\xi}{\xi - z} \quad (16)$$

For $\omega(\xi) = \phi + i\psi$ and by assumption $\phi = \alpha_\phi(\bar{\phi}, \xi)$ for $\xi \in \Gamma$, (16) obviously simplifies to:

$$2\pi e(z_1^-) = \lim_{z \rightarrow z_1^-} \int_{\Gamma} \frac{e_\psi d\xi}{\xi - z}, \quad e_\psi \equiv (\psi - \alpha_\psi) \quad (17)$$

Because ϕ is known continuously on Γ , trial functions can be selected to provide within arbitrary accuracy $\alpha_\phi = \phi$ on each Γ_j . Consequently, the total relative error $e(z)$ can be assumed generated entirely from the e_ψ function. If the problem domain is linearly translated such that z_1^- coincides with the origin, then (17) simplifies to:

$$2\pi i e(z_1^-) = \lim_{z \rightarrow 0^-} \int_{\Gamma} \frac{e_\psi d\xi}{\xi - z} \quad (18)$$

The magnitude of nodal value $e_1 \equiv e(z_1^-)$ is given by:

$$2\pi |e_1| = \lim_{z \rightarrow 0^-} \left| \int_{\Gamma} \frac{e_\psi d\xi}{\xi - z} \right| < \lim_{z \rightarrow 0^-} \left| \int_{\Gamma_1 \cup \Gamma_m} \frac{e_\psi d\xi}{\xi - z} \right|$$

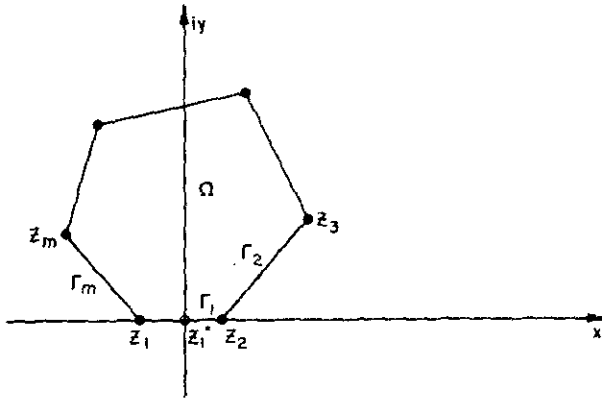


Figure 2. Translation of Γ to coincide point z_1^* to origin of axis

$$+ \sum_{j=2}^{m-1} \left| \int_{\Gamma_j} \frac{e_\psi d\zeta}{\zeta} \right| \quad (19)$$

where Γ_1 and Γ_m share nodal co-ordinate z_1 and, therefore, are involved in the Cauchy principal value evaluation.

The $|e(z)|$ function can be evaluated at a point z_1^* interior of an element Γ_1 similar to (19) where the boundary is assumed translated so that z_1^* coincides with the origin (Fig. 2), by:

$$2\pi|e_1^*| = \lim_{z \rightarrow 0^+} \left| \int_{\Gamma} \frac{e_\psi d\zeta}{\zeta - z} \right| \leq \lim_{z \rightarrow 0^+} \left| \int_{\Gamma_1} \frac{e_\psi d\zeta}{\zeta - z} \right| + \sum_{j=2}^m \left| \int_{\Gamma_j} \frac{e_\psi d\zeta}{\zeta} \right| \quad (20)$$

The problem of (20) will be studied in the following (with (19) being analogous in development). Consider the integral on element $\Gamma_k, k \neq 1$ by:

$$\left| \int_{\Gamma_k} \frac{e_\psi d\zeta}{\zeta} \right| \leq \int_{\Gamma_k} \frac{|e_\psi| |d\zeta|}{|\zeta|} \leq M_k \frac{|\Gamma_k|}{\mathcal{D}_k} \quad (21)$$

where

$$M_k = \max |e_\psi(z)|, \quad z \in \Gamma_k$$

$$\mathcal{D}_k = \min |z|, \quad z \in \Gamma_k$$

On arbitrary element Γ_1 , select the study point z_1^* such that $|e(z_1^*)|$ is a maximum with $|e(z_1^*)| = M_1$. Then:

$$2\pi M_1 \leq \lim_{z \rightarrow 0^+} \left| \int_{\Gamma_1} \frac{e_\psi d\zeta}{\zeta - z} \right| + \sum_{j=2}^m M_j \frac{|\Gamma_j|}{\mathcal{D}_j} \quad (22)$$

Equation (22) applies to each boundary element for the case of M_1 occurring interior of each Γ_1 .

From Fig. 3, point z_1^* (origin) is interior of Γ_1 of length $|\Gamma_1| = L = L_a + 2\epsilon + L_b$. Thus:

$$\lim_{z \rightarrow 0^+} \int_{\Gamma_1} \frac{e_\psi d\zeta}{\zeta - z} = \int_{-L_a - \epsilon}^{-\epsilon} \frac{e_\psi d\zeta}{\zeta} + \lim_{z \rightarrow 0^+} \int_{-\epsilon}^{\epsilon} \frac{e_\psi d\zeta}{\zeta - z}$$

$$+ \int_{\epsilon}^{L_b + \epsilon} \frac{e_\psi d\zeta}{\zeta} \quad (23)$$

The integrand term $d\zeta/\zeta$ is monotonic on the closed intervals $[-L_a - \epsilon, -\epsilon]$ and $[\epsilon, L_b + \epsilon]$ and can be evaluated in magnitude by:

$$\left| \int_{-L_a - \epsilon}^{-\epsilon} \frac{e_\psi d\zeta}{\zeta} \right| + \left| \int_{\epsilon}^{L_b + \epsilon} \frac{e_\psi d\zeta}{\zeta} \right| \leq M_1 \left(\ln \frac{L_a + \epsilon}{\epsilon} + \ln \frac{L_b + \epsilon}{\epsilon} \right) \quad (24)$$

where it is assumed that $L_a L_b > 0$. In (24), the maximum value of the logarithm sum is when $L_a = L_b = |\Gamma_1|/2 - \epsilon$. The remaining integral term of (23) is analyzed by letting $e_\psi(z) = e'_\psi + f(z)$ in the closed interval $[-\epsilon, \epsilon]$ where e'_ψ is an assumed mean value of $e_\psi(z)$ in $[-\epsilon, \epsilon]$. Then:

$$\lim_{z \rightarrow 0^+} \int_{-\epsilon}^{\epsilon} \frac{e_\psi d\zeta}{\zeta - z} = \lim_{z \rightarrow 0^+} e'_\psi \int_{-\epsilon}^{\epsilon} \frac{d\zeta}{\zeta - z} + \lim_{z \rightarrow 0^+} \int_{-\epsilon}^{\epsilon} \frac{f(\zeta) d\zeta}{\zeta - z} \quad (25)$$

From (24) and (25) and setting $L_a = L_b = |\Gamma_1|/2 - \epsilon$:

$$\left| \lim_{z \rightarrow 0^+} \int_{\Gamma_1} \frac{e_\psi d\zeta}{\zeta - z} \right| \leq 2M_1 \ln \left(\frac{L_a + \epsilon}{\epsilon} \right) + E_1 \quad (26)$$

where

$$E_1 = \left| \lim_{z \rightarrow 0^+} \int_{-\epsilon}^{\epsilon} \frac{f(\zeta) d\zeta}{\zeta - z} \right|$$

Substituting (26) into (22) gives a generalized relative error bound for each boundary element Γ_1 :

$$2\pi M_1 \leq 2M_1 \ln \left(\frac{|\Gamma_1|}{2\epsilon} \right) + \sum_{j=2}^m M_j \frac{|\Gamma_j|}{\mathcal{D}_j} + E_1 \quad (27)$$

Equation (27) can be studied for the simplifying assumption of E_1 being assumed zero for each Γ_1 .

It is convenient to assume that the E_1 term is negligible for the special case of:

$$\epsilon = \frac{1}{2} |\Gamma_1| \exp\left(\frac{1}{2} - \pi\right) \sim 0.0356 |\Gamma_1| \quad (28)$$

Then (27) simplifies to:

$$M_1 \leq \sum_{j=2}^m M_j \frac{|\Gamma_j|}{\mathcal{D}_j} \quad (29)$$

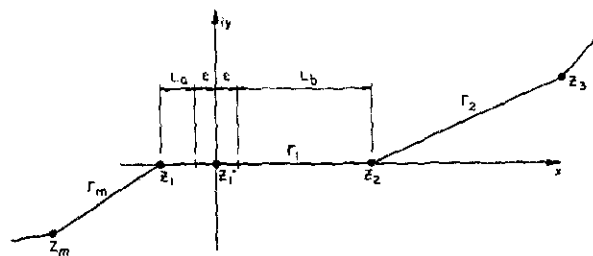


Figure 3. Principal value integral evaluation geometry

The results of (29) can be easily used in the CVBEM modeling approach to analyze the sensitivity of relative error reduction with respect to error contributions from various locations along the problem boundary.

CONVERGENCE OF THE CVBEM

Convergence of the CVBEM approximation function $\hat{\omega}(z)$ to the analytic boundary value problem solution $\omega(z)$ (if $\omega(z)$ exists) is tested by verifying:

$$\lim_{\delta \rightarrow 0} \hat{\omega}(z) = \omega(z), \quad z \in \Gamma \cup \Omega \quad (30)$$

where $\delta = |\Gamma_j|$; that is, Γ is now subdivided into equilength elements. A straightforward approach to this problem is to show that:

$$\lim_{\delta \rightarrow 0} \hat{\psi} = \psi$$

(where in our study case ϕ is the known boundary conditions on Γ) and then conclude that by the Cauchy-Riemann equations, $\hat{\omega}(z) = \omega(z)$ in the limit.

From (12), a method to determine the unknown nodal values of $\hat{\psi}$ on Γ is to set:

$$\hat{\psi} = C_1(\bar{\phi}, \hat{\psi})$$

which results in an approximation function $\hat{\omega}$ such that at nodal point j ($z_j \in \Gamma$):

$$\hat{\omega}_j = \hat{\phi}_j + i\hat{\psi}_j = \hat{\phi}_j + i\bar{\psi}_j$$

Thus for m nodal points on Γ :

$$\hat{\psi}(z_j) = \bar{\psi}(z_j), \quad j = 1, 2, 3, \dots, m$$

The true solution of the boundary value problem, $\omega(z)$, is analytic on $\Omega \cup \Gamma$ and is therefore uniformly continuous on the closed set $\Omega \cup \Gamma$. The approximator $\hat{\omega}(z)$ is also analytic on Ω for all $z \notin \Gamma$ and is therefore continuous in that region. On the boundary Γ , however, $\hat{\omega}(z)$ is not analytic but the Cauchy principal value of the limit exists as $\hat{\omega}(z_0)$ is evaluated for $z_0 \in \Gamma$ by:

$$2\pi i \hat{\omega}(z_0) \equiv \lim_{z \rightarrow z_0} \int_{\Gamma} \frac{[\alpha_{\phi} + i\alpha_{\psi}] d\xi}{\xi - z} \quad z_0 \in \Gamma \quad (31)$$

where $\hat{\omega}(z_0)$ is defined to be the above limiting value (as z approaches $z_0 \in \Gamma$ from the interior of Ω).

The approximation integral function of (31) would equal the true solution $\omega(z)$ if the trial functions for the unknown ψ variable were the solution to the boundary value problem. That is, the linear trial function $\alpha_{\psi}(\psi, \xi)$ can be corrected by a continuous function $\alpha^*(\xi)$ such that (Fig. 4):

$$\psi(\xi) = \alpha_{\psi}(\psi_j, \xi) + \alpha^*(\xi), \quad \alpha^*(z_j) = 0 \quad (32)$$

where the true nodal values ψ_j are now used in the approximation. Thus for $\alpha_{\psi}(\phi_j, \xi) = \phi(\xi)$ on Γ :

$$2\pi i \omega(z) = \int_{\Gamma} \frac{[\alpha_{\phi}(\phi_j, \xi) + i\alpha_{\psi}(\psi_j, \xi) + i\alpha^*(\xi)] d\xi}{\xi - z} \quad (33)$$

In (33), $\alpha^*(\xi)$ identically zero on Γ implies $\hat{\omega}(z) \equiv \omega(z)$. Because $\psi(\xi)$ is uniformly continuous on the simple closed contour Γ , then for every $\epsilon' > 0$ there exists a $\delta > 0$ such that $|\Gamma_j| < \delta$ guarantees that $|\alpha^*(\xi)| < \epsilon'$. Rewriting (33) as:

$$2\pi i \omega(z) = \int_{\Gamma} \frac{[\alpha_{\phi}(\phi_j, \xi) + i\alpha_{\psi}(\psi_j, \xi)] d\xi}{\xi - z} + \int_{\Gamma} \frac{\alpha^*(\xi) d\xi}{\xi - z} \quad (34)$$

$$\alpha^*(\xi) = \psi(\xi) - \alpha_{\psi}(\psi_j, \xi)$$

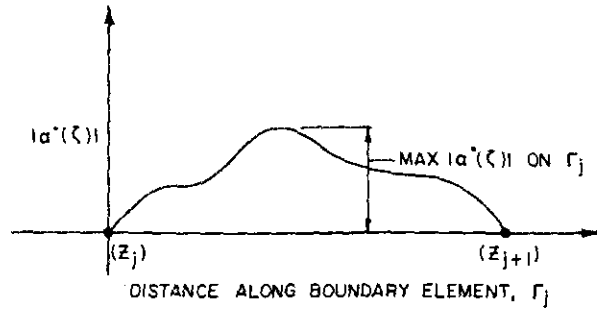


Figure 4. Relationship of correction function $\alpha^*(\xi)$ to the function $\psi(\xi)$

the correction function integral can be evaluated as:

$$\int_{\Gamma} \frac{\alpha^*(\xi) d\xi}{\xi - z} = \sum_{j=1}^m \int_{\Gamma_j} \frac{\alpha^*(\xi) d\xi}{\xi - z} \quad (35)$$

For $z_1 \notin \Gamma$, let $R = \min|z - z_1|$, $z \in \Gamma$. Then choose a δ such that $\epsilon' = \epsilon R / |\Gamma|$ where $\epsilon > 0$ is the desired tolerance. Then:

$$\left| \int_{\Gamma} \frac{\alpha^*(\xi) d\xi}{\xi - z_1} \right| \leq \sum_{j=1}^m \int_{\Gamma_j} \frac{|\alpha^*(\xi)| |d\xi|}{|\xi - z_1|} < \frac{m \epsilon R \delta}{|\Gamma| R} = \epsilon \quad (36)$$

where $m = |\Gamma|/\delta$.

Thus as $\delta = |\Gamma_j|$ approaches 0 in the limit the contribution from the correction trial function becomes arbitrarily small and, therefore, $\hat{\omega}(z)$ approaches $\omega(z)$ point wise. For $z_1 \in \Gamma_1$, a similar argument to (36) is made except (dropping limit notation):

$$\left| \int_{\Gamma} \frac{\alpha^*(\xi) d\xi}{\xi - z_1} \right| \leq \left| \int_{\Gamma_1} \frac{\alpha^*(\xi) d\xi}{\xi - z_1} \right| + \sum_{j=2}^m \int_{\Gamma_j} \frac{|\alpha^*(\xi)| |d\xi|}{|\xi - z_1|} \quad (37)$$

Letting $R = \min|z - z_1|$, $z \in \Gamma$ and $z \notin \Gamma_1$ (and assuming z_1 is the midpoint of Γ_1), then choose a $\delta = |\Gamma_j|$, $j \neq 1$, such that $\epsilon' = \epsilon_2 R / |\Gamma|$. Then analogous to the development of (22) and (36):

$$\left| \int_{\Gamma} \frac{\alpha^*(\xi) d\xi}{\xi - z_1} \right| < \left| \int_{\Gamma_1} \frac{\alpha^*(\xi) d\xi}{\xi - z_1} \right| + \frac{(m-1) \epsilon_2 R \delta}{|\Gamma| R} < \epsilon_1 + \epsilon_2 < \epsilon$$

where $R = |\Gamma_1|/2$ is chosen such that the Cauchy principal value:

$$\left| \int_{\Gamma_1} \frac{\alpha^*(\xi) d\xi}{\xi - z_1} \right| < \epsilon_1$$

APPLICATIONS

The relative error bound relationship of (29) (or more precisely from (27)) can be used to evaluate the response of the CVBEM model by adding an additional nodal point to Γ .

I. Suppose an additional nodal point z_a is added to the midpoint of boundary element Γ_m . From (29):

$$M_1 \leq \sum_{j=2}^{m-1} M_j \frac{|\Gamma_j|}{\mathcal{D}_j} + M_m \frac{|\Gamma_m|}{\mathcal{D}_m} \quad (39)$$

But for z_a being a new midpoint node on Γ_m implies:

$$M_m \frac{|\Gamma_m|}{\mathcal{D}_m} \geq M_a \frac{|\Gamma_a|}{\mathcal{D}_a} + M_b \frac{|\Gamma_b|}{\mathcal{D}_b} \quad (40)$$

where $\Gamma_a \cup \Gamma_b = \Gamma_m$, and the other terms follow similarly. Since $\mathcal{D}_m = \min(\mathcal{D}_a, \mathcal{D}_b)$ and $M_m = \max(M_a, M_b)$, it can be concluded from geometric reasoning alone that adding another nodal point will reduce the relative error bound (or result in no change).

II. Another method to analyzing bounds on the relative error norm is to assume a trial function distribution of the $e_\psi(z)$ error function on arbitrary element Γ_1 . One approach is to let $e_\psi(z) \in \{P_n(z), z \in \Gamma_1\}$ where $P_n(z)$ is an n th order polynomial of $(n+1)$ nodal values (equally spaced) on Γ_1 :

$$P_n(z) = \sum_{k=0}^n N_k(z) e_k \quad (41)$$

where $N_k(z)$ is the usual n th order polynomial shape function and e_k is a nodal point value of e_ψ for $z_k \in \Gamma_1$.

For $n=0$ and $e_\psi \equiv e'_\psi$ (a constant on Γ_1):

$$\lim_{z \rightarrow 0^+} \int_{\Gamma_1} \frac{e_\psi d\xi}{\xi - z} = 0$$

For $n=1$:

$$e_\psi = e_1 \left(\frac{z_{j+1} - z}{z_{j+1} - z_j} \right) + e_2 \left(\frac{z - z_j}{z_{j+1} - z_j} \right)$$

and

$$\left| \lim_{z \rightarrow 0^+} \int_{\Gamma_1} \frac{e_\psi d\xi}{\xi - z} \right| = |e_2 - e_1| \leq 2M_1$$

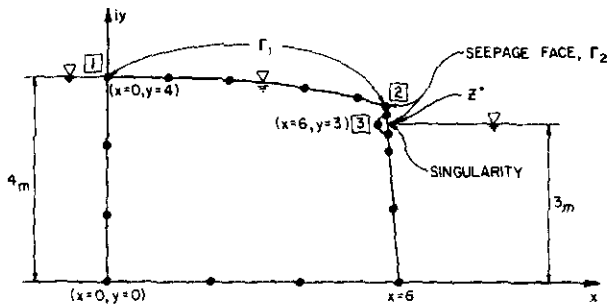


Figure 5. CVBEM nodal placement for seepage problem and modeling relative error

Similarly for $n=3$, the cubic variation of e_ψ on Γ_1 results in a maximum relative error norm of:

$$\left| \lim_{z \rightarrow 0^+} \int_{\Gamma_1} \frac{e_\psi d\xi}{\xi - z} \right| = \left| -\frac{e_1}{4} - \frac{9e_2}{4} + \frac{9e_3}{4} + \frac{e_4}{4} \right| \leq 5M_1$$

where $\{e_1, e_2, e_3, e_4\}$ are evenly spaced nodal values of $e_\psi(z)$ for $z \in \Gamma_1$. It can be noted that the bound on the relative error norm becomes arbitrarily large as the trial function polynomial order becomes arbitrarily large.

III. Should a conformal mapping transformation be found which maps $\Omega \cup \Gamma$ into the unit circle C and its interior, and the solution to the boundary value problem is analytic on C and its simply connected interior, then the analysis of relative error determination is simplified. Letting C be defined by $\{z: |z|=1\}$, then z can be described by $z = Re^{i\theta}$, where $R \leq 1$ and $0 \leq \theta < 2\pi$. In this application, C is assumed discretized into $2(n+1)$ equilength boundary elements of length $|\Gamma_j| = \eta$. If the trial function is assumed to be a polynomial as a function of all nodal values $\tilde{\omega}_j$, then the integral approximation function reduces to an n -order complex polynomial centered at the origin of the unit circle:

$$\tilde{\omega}(z) = \frac{1}{2\pi i} \int_C \frac{[\alpha_\phi + i\alpha_\psi] d\xi}{\xi - z} = \sum_{j=0}^n a_0 z^n \quad (42)$$

where $a_k = (\alpha_k + i\beta_k)$; $\{\alpha_k, \beta_k\} \in \mathcal{R}$.

Again assuming $\phi(R=1, \theta)$ is known on C , then $P_n(z)$ can be constructed by setting:

$$\sum_{j=0}^n (\alpha_j + i\beta_j) \exp(ij\theta) R^j = \tilde{\omega}(\theta) \quad (43)$$

which reduces to the real and imaginary formulas of:

$$\begin{aligned} &\alpha_0 + R(\alpha_1 \cos \theta - \beta_1 \sin \theta) + \dots \\ &+ R^n(\alpha_n \cos n\theta - \beta_n \sin n\theta) = \phi(R, \theta) \\ &\beta_0 + R(\alpha_1 \sin \theta + \beta_1 \cos \theta) + \dots \\ &+ R^n(\alpha_n \sin n\theta + \beta_n \cos n\theta) = \psi(R, \theta) \end{aligned} \quad (44)$$

The (α_k, β_k) are determined for $P_n(z)$ by solving (44) for the $2(n+1)$ known values of ϕ (and at least one value of ψ is known) on C . Because $P_n(z)$ is of finite order it is uniformly continuous and analytic on $C \cup \Omega$. Relative error is easily integrable by noting:

$$e(z) \equiv \omega(R, \theta) - \tilde{\omega}(R, \theta) = (\phi - \hat{\phi}) + i(\psi - \hat{\psi})$$

and $e(z)$ is analytic inside and on C giving:

$$\int_C e(z) dz = 0 \quad (45)$$

which implies:

$$\left| \int_C (\phi - \hat{\phi}) dz \right| = \left| \int_C (\psi - \hat{\psi}) dz \right| \quad (46)$$

In this application it is assumed that $\phi(R=1, \theta)$ is known on C and, consequently, $\phi_e = \phi - \hat{\phi}$ is known continuously on C . Thus, $\phi_e = (\phi - \hat{\phi})$ on C where C is subdivided (initially) by $2(n+1)$ equidistant nodal points z_j where $\phi_e(z_j) = 0$.

Of interest is the effect of nodal point density on C . Again using the relative error norm, the objective is to

minimize ϕ_e on each Γ_j . Since each element Γ_j is a straight line segment between co-ordinates z_j and z_{j+1} (both on C), the x - y co-ordinate pairs are related to a first order polynomial parametric function of variable t . Thus on:

$$\Gamma_k \equiv \{z: z = z_k(1-t) + z_{k+1}t, 0 \leq t \leq 1\}$$

$x = x(t)$ and $y = y(t)$. The maximum value of $\bar{\phi}_e(k)$ on Γ_k is determined by differentiation of ϕ_e on each Γ_k (a maximum $\bar{\phi}_e(k)$ must be interior of Γ_k as $\phi_e = 0$ at nodal points). Given the element locations of $\bar{\phi}_e(k)$, two additional nodal points are added at those locations on C where two $\bar{\phi}_e(k)$ are maximum (two points are needed due to the two-part complex coefficient).

Thus, after an additional two nodal points are added to C , a new discretization (or partition) P^* is formed where the original partition P is a subset of P^* . From (27), $\phi_e(P^*) \leq \phi_e(P)$.

CONCLUSIONS

In this paper, convergence of the CVBEM to the true solution of the boundary value problem is examined. Addi-

tionally, bounds on the relative error norm are determined and a straightforward relationship for evaluating the inter-element dependence on relative error is forwarded. Using the developed relationships will provide an easy to apply test for selecting subsequent nodal point densities in order to reduce the relative error norm on any boundary element.

REFERENCES

- 1 Hromadka II, T. V. and Guymon, G. L. A complex variable boundary element method: development, *Int. J. of Numerical Methods in Engineering* 1984, 20, 25-37
- 2 Hromadka II, T. V. and Guymon, G. L. An algorithm to reduce approximation error from the complex variable boundary element method applied to soil freezing, submitted to *Numerical Heat Transfer* 1983 (in press)
- 3 Hunt, B. and Isaacs, L. Integral equation formulation for groundwater flow, *J. of the Hydraulic Division, ASCE* 1981, 107 (HY110), Proc. Paper 16539, 1197-1209
- 4 Hromadka II, T. V. and Guymon, G. L. Application of a boundary integral equation to prediction of freezing fronts in soil, *Cold Regions Science and Technology* 1982, 6, 115-121