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A COMPLEX VARIABLE BOUNDARY ELEMENT METHOD: DEVELOPMENT

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SUMMARY

A generalized boundary integral equation method for the solution of the Laplace equation is developed based on the Cauchy integral theorem for analytical complex variable functions. Although the approach is complicated by the utilization of complex variable theory, the resulting model involves direct integration along straight-line boundary segments (elements) rather than using quadrature formulae that are required in current real variable boundary element formulations. Previously published boundary integral equation methods based on the Cauchy integral theorem are shown to be a subset of the generalized model theory developed in this paper.

INTRODUCTION

Many engineering problems such as potential flow and seepage are governed by the well-known Laplace equation. One of the techniques used in approximately solving potential flow problems is the Boundary Integral Equation Method (BIEM). The solutions of the BIEM are generally based on approximation real variable functions that satisfy the Laplace equation. Free space Green's function and Green's theorem are widely used to determine a statement of the governing partial differential equation. Butterfield and Tomlin[1] and Butterfield[2] used a free space Green's function to study potential flow problems in anisotropic continuum, homogeneous and inhomogeneous bodies, respectively. Walker[3] applied Green's function in analysing the fluid–structure interaction problems. Groenenbroom[4] used the Green's function and the Kirchhoff method to study the steady and unsteady potential flow of a compressible liquid inside a fixed volume.

An alternative to the Green's function solution is the fundamental solution approach. This technique chooses a real variable function that satisfies the domain condition of the problem, but does not necessarily satisfy the boundary conditions. This fundamental solution is used with Green's second identity to obtain a general method for solving potential flow problems. This approach was used by Brebbia and Dominguez,[5] Brebbia,[6] Brebbia and Nakaguma,[7] Brebbia and Wrobel,[8] Bratanow *et al.*,[9] Brebbia and Walker[10] and Tanaka and Tanaka[11] and Lennon *et al.*[12] Recently, Söhngen and Bischoff[13] applied Green's third equation to the real potential functions (U and $\ln(r)$), and by using the Cauchy–Riemann differential equations for the orthogonal functions, $\ln(r)$ and angle β, an integral equation of the first kind for the potential function U was obtained. An application of a BIEM to a field flow problem is given in Lennon *et al.*[12]

A BIEM model, based on the Cauchy integral theorem using a linear trial function between boundary nodal points, was developed by Hunt and Isaacs,[14] to study steady-state potential flow problems. Hromadka and Guymon[15] extended this model to accommodate moving boundary problems.

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In this paper, a generalization of the real variable BIEM modelling approach to the complex variable system is made. The BIEM is based on the Cauchy integral theorem, but in this paper the trial function approximation is extended to an arbitrary polynomial order. Additionally, flux-type boundary conditions are accommodated in the model. Integration formulae are derived which simplify the development of a complex variable BIEM model.

BIEM APPROXIMATION

A complex variable analytic function $\omega(z)$ is composed of two real variable two-dimensional functions:

$$\omega(z)=\phi(x,y)+i\psi(x,y), \qquad z\in\hat{\Omega} \tag{1}$$

where $z=x+iy$; $i=\sqrt{-1}$; $\phi(x,y)$ is a potential function; and $\psi(x,y)$ is a stream function. In equation (1), $\omega(z)$ is only defined in a simply-connected domain $\hat{\Omega}$ wherein $\omega(z)$ is analytic.[16] The real variable functions composing $\omega(z)$ are related by the Cauchy–Riemann equations in $\hat{\Omega}$:

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y}, \qquad \frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x} \tag{2}$$

Consequently, each function is harmonic and satisfies the Laplace equation

$$\frac{\partial^2\xi}{\partial x^2}+\frac{\partial^2\xi}{\partial y^2}=0; \qquad z\in\hat{\Omega}, \qquad \xi=\phi,\psi \tag{3}$$

For an analytic function $\omega(z)$ defined in domain $\hat{\Omega}$ and on boundary $\hat{\Gamma}$ (Figure 1), Cauchy's integral theorem equates the value of $\omega(z_0)$ to a boundary integral on $\hat{\Gamma}$ with

$$2\pi i\omega(z_0)=\oint_{\hat{\Gamma}}\frac{\omega(z)\,\mathrm{d}z}{z-z_0} \tag{4}$$

where z_0 is interior of $\hat{\Omega}$, and the line integral integration is in the positive sense (Figure 2). From equation (4), a BIEM can be developed which is analogous to current real variable BIEM models. Similar to the real variable BIEM models, the problem domain $\hat{\Omega}$ and boundary $\hat{\Gamma}$ is redefined as a global domain Ω and boundary Γ, as shown in Figure 3. The global boundary Γ is then subdivided into straight-line segments by boundary nodal points.

In the complex variable model, each boundary segment can contain one or more interior nodal points. Consequently, a different approximation function can be arbitrarily described

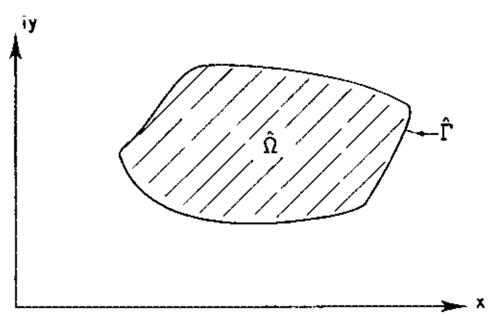

Figure 1. Problem domain $\hat{\Omega}$ with boundary contour $\hat{\Gamma}$

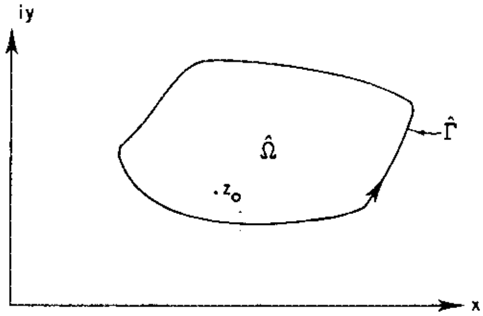

Figure 2. Domain $\hat{\Omega}$ with interior point z_0. Positive sense direction on boundary $\hat{\Gamma}$ is shown by arrowhead

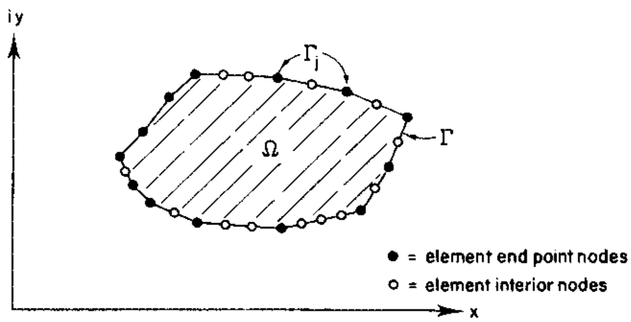

Figure 3. Global domain Ω with global boundary Γ subdivided into boundary elements Γ_j

on each boundary segment in a fashion similar to using various one-dimensional finite elements in a finite element domain numerical solution.[17] Because each boundary segment approximation function may be independent from neighbouring boundary segment approximation functions, except for continuity at shared nodal points, the boundary segments will be called Complex Variable Boundary Elements (CVBE) to differentiate the method from the real variable Boundary Element (BEM) procedures.[7,10]

For a point z_0 interior of Ω, the Cauchy integral can be rewritten in terms of m-CVBE by

$$2\pi i\omega(z_0)=\sum_{j=1}^{m}\int_{\Gamma_j}\frac{\omega(z)\,\mathrm{d}z}{z-z_0} \tag{5}$$

where Γ_j is a CVBE in a m-element model. On each Γ_j, n-nodal points $(z_1, z_2, \ldots, z_n)$ are located such as shown in Figure 4. By defining a linear local co-ordinate system (Figure 5), the following relationships can be used to calculate the CVBE Γ_j contribution to the line integral of equation (5):

$$\left.\begin{aligned} z(s)&=z_1+(z_n-z_1)s\\ \mathrm{d}z&=(z_n-z_1)\,\mathrm{d}s\\ \omega(z)&=\omega(z(s)) \end{aligned}\right\} z\in\Gamma_j, \qquad 0\leqslant s\leqslant 1 \tag{6}$$

In the following, $\omega(s)$ will be used as notation for the $\omega(z(s))$ function on each Γ_j.

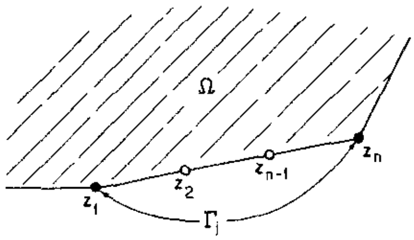

Figure 4. Boundary element Γ_j with associated nodal points at locations z_1–z_n

The Cauchy boundary integral can be written as the sum of m-CVBE contributions

$$2\pi i\omega(z_0)=\sum_{j=1}^{m}\left(\int_{s=0}^{1}\frac{\omega(s)(z_n-z_1)\,\mathrm{d}s}{z(s)-z_0}\right)_j \tag{7}$$

where the subscript $(\ )_j$ notation indicates values from CVBE Γ_j. Equation (7) can be simplified to

$$2\pi i\omega(z_0)=\sum_{j=1}^{m}\left(\int_{s=0}^{1}\frac{\omega(s)\,\mathrm{d}s}{s+\mathscr{C}}\right)_j, \qquad 0\leqslant s\leqslant 1 \tag{8}$$

where

$$\mathscr{C}=\left(\frac{z_1-z_0}{z_n-z_1}\right)_j \tag{9}$$

and

$$\omega(s)=\phi(s)+i\psi(s), \qquad 0\leqslant s\leqslant 1 \tag{10}$$

Equation (8) can now be rewritten as a sum of the complex variable definite integrals

$$\left(\int_{s=0}^{1}\frac{\omega(s)\,\mathrm{d}s}{s+\mathscr{C}}\right)_j=\left(\int_{s=0}^{1}\frac{\phi(s)\,\mathrm{d}s}{s+\mathscr{C}}\right)_j+i\left(\int_{s=0}^{1}\frac{\psi(s)\,\mathrm{d}s}{s+\mathscr{C}}\right)_j \tag{11}$$

The assumed approximation functions for $\phi(s)$ and $\psi(s)$ are substituted into equation (11), where

$$\phi(s)\equiv\sum_{k=1}^{n}N_k(s)\phi_k, \qquad 0\leqslant s\leqslant 1 \tag{12}$$

$$\psi(s)\equiv\sum_{k=1}^{n}N_k(s)\psi_k, \qquad 0\leqslant s\leqslant 1 \tag{13}$$

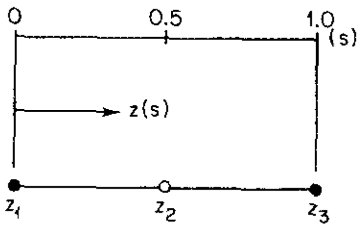

Figure 5. Boundary element local co-ordinate system definition

The shape functions $N_k(s)$ are analogous to the one-dimensional interpolation functions used in finite element methods;[17] and (ϕ_k, ψ_k) indicate values of ϕ and ψ at nodal point k of CVBE Γ_j.

Combining equations (8), (12) and (13) gives the BIEM approximation statement

$$2\pi i\omega(z_0) = \sum_{j=1}^{m}\left(\int_{s=0}^{1}\frac{\sum N_k\phi_k\,\mathrm{d}s}{s+\mathscr{C}}\right)_j + i\sum_{j=1}^{m}\left(\int_{s=0}^{1}\frac{\sum N_k\psi_k\,\mathrm{d}s}{s+\mathscr{C}}\right)_j \tag{14}$$

BIEM BOUNDARY NODE APPROXIMATION

Analogous to real variable BIEM modelling, either values of ϕ or ψ are known along each CVBE. Therefore, the unknown real function value needs to be determined for each nodal point defined on boundary Γ. An equation can be developed which defines an unknown nodal value as a function of all remaining unknown nodal values by calculating the boundary integral in equation (14) as interior point z_0 approaches each boundary node z_i in turn. This limit process results in a Cauchy principal value of the integral.

Two techniques of determining the principal value will be presented. One approach[16] is to redefine the global boundary at node i with a circle Γ_ε of epsilon radius as shown in Figure 6, where

$$\Gamma_\varepsilon: z - z_i = \varepsilon\,\mathrm{e}^{i\zeta}, \qquad 0 \leqslant \zeta \leqslant (2\pi - \theta) \tag{15}$$

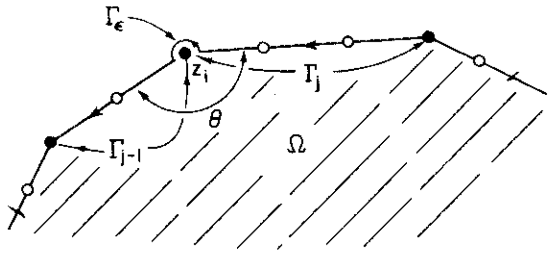

Figure 6. Definition of contour integration for Cauchy principal value determination

Therefore the Cauchy principle value is determined by

$$2\pi i \lim_{\varepsilon\to 0}\omega(z_i) = \lim_{\varepsilon\to 0}\int_{\Gamma'+\Gamma_\varepsilon}\frac{\omega(z)\,\mathrm{d}z}{z - z_i} \tag{16}$$

where Γ' is the global boundary Γ with that portion interior of a complete circle centred at point z_i (with radius ε) deleted. Expanding equation (16),

$$\lim_{\varepsilon\to 0}\int_{\Gamma'}\frac{\omega(z)\,\mathrm{d}z}{z - z_i} = \int_{\Gamma}\frac{\omega(z)\,\mathrm{d}z}{z - z_i} \tag{17}$$

and from equation (15)

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon}\frac{\omega(z)\,\mathrm{d}z}{z - z_i} = \omega(z_i)(2\pi - \theta)i \tag{18}$$

where angle θ is as shown in Figure 6. From the above, the BIEM approximation statement for boundary points $z_i \in \Gamma$ is given by the principal values

$$\theta i\omega(z_i) = \sum_{j=1}^{m} \left(\int_{s=0}^{1} \frac{\sum N_k \phi_k \, ds}{s+\mathscr{C}} \right)_j + i \sum_{j=1}^{m} \left(\int_{s=0}^{1} \frac{\sum N_k \psi_k \, ds}{s+\mathscr{C}} \right)_j \tag{19}$$

where for element interior nodal points, $\theta = \pi$.

A second method to determine the Cauchy principal value will be presented by means of an example (the results which will be used in a later section). Consider a linear trial function between successive nodes (z_1, z_2) and (z_2, z_3). Then for $z_0 \in \Omega$ and $z_0 \notin \Gamma$ (Figure 7),

$$2\pi i\omega(z_0) = \int_{z_1}^{z_2} \frac{\omega(\zeta)\,d\zeta}{\zeta - z_0} + \int_{z_2}^{z_3} \frac{\omega(\zeta)\,d\zeta}{\zeta - z_0} - \sum_{z_i \neq z_1, z_2} \int_{z_i}^{z_{i+1}} \frac{\omega(\zeta)\,d\zeta}{\zeta - z_0} \tag{20}$$

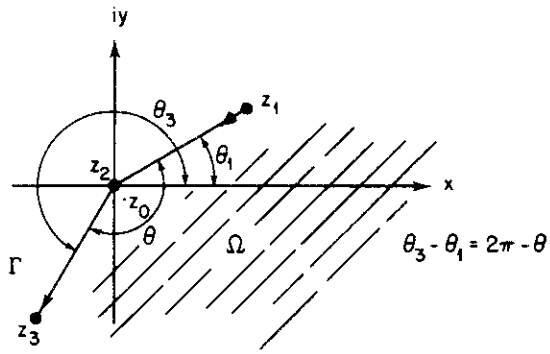

Figure 7. Cauchy principal value parameter definitions

Define

$$I = 2\pi i\omega(z_0) - \int_{z_1}^{z_2} \frac{\omega(\zeta)\,d\zeta}{\zeta - z_0} - \int_{z_2}^{z_3} \frac{\omega(\zeta)\,d\zeta}{\zeta - z_0} \tag{21}$$

For the linear trial functions assumptions,

$$I = 2\pi i\omega(z_0) - \left[\omega_2 - \omega_1 + \omega_1 \left(\frac{z_2 - z_0}{z_2 - z_1} \right) \ln\left(\frac{z_2 - z_0}{z_1 - z_0} \right) - \omega_2 \left(\frac{z_1 - z_0}{z_2 - z_1} \right) \ln\left(\frac{z_2 - z_0}{z_1 - z_0} \right) \right]$$
$$- \left[\omega_3 - \omega_2 + \omega_2 \left(\frac{z_3 - z_0}{z_3 - z_2} \right) \ln\left(\frac{z_3 - z_0}{z_2 - z_0} \right) - \omega_3 \left(\frac{z_2 - z_0}{z_3 - z_2} \right) \ln\left(\frac{z_3 - z_0}{z_2 - z_0} \right) \right] \tag{22}$$

In the limit as z_0 approaches z_2, $(z_0 \in \Omega,\ z_2 \in \Gamma)$,

$$\lim_{z_0 \to z_2} I = 2\pi i\omega(z_2) - \left[(\omega_3 - \omega_1) + \omega_2 \ln\left(\frac{z_3 - z_2}{z_1 - z_2} \right) \right] \tag{23}$$

Simplifying,

$$\lim_{z_0 \to z_2} I = 2\pi i\omega_2 - (\omega_3 - \omega_1) + \omega_2 \ln\left| \frac{z_1 - z_2}{z_3 - z_2} \right| - i(2\pi - \theta)\omega_2 \tag{24}$$

Hence,

$$\omega_1-\omega_3+\omega_2\left[\ln\left|\frac{z_1-z_2}{z_3-z_2}\right|+i\theta\right]=\sum_{z_i\neq z_1,z_2}\int_{z_i}^{z_{i+1}}\frac{\omega(\zeta)\,\mathrm{d}\zeta}{\zeta-z_0} \tag{25}$$

CVBE INTEGRATION CONTRIBUTIONS

On each CVBE Γ_j, an approximation function is defined such that for n-nodal points on Γ_j

$$\omega(s)=\sum_{k=1}^{n}N_k(s)\omega_k,\qquad 0\leq s\leq 1 \tag{26}$$

where $\omega_k=\phi_k+i\psi_k$. A convenient family of interpolation functions is the usual set of polynomials where

$$N_k(s)=\begin{cases}1, & z=z_k\\ 0, & z=z_j, \qquad j\neq k\end{cases} \tag{27}$$

Then for a $(n+1)$-nodal point element,

$$N_k(s)=a_0+a_1s+\ldots+a_ns^n,\quad 0\leq s\leq 1 \tag{28}$$

For element Γ_j with $z_0\notin\Gamma_j$ the integration contribution to the boundary integral may be calculated from the relations

$$\begin{aligned}
&\int\frac{\mathrm{d}s}{s+\mathscr{C}}=\ln(s+\mathscr{C})\\
&\int\frac{s\,\mathrm{d}s}{s+\mathscr{C}}=s-\mathscr{C}\ln(s+\mathscr{C})\\
&\int\frac{s^2\,\mathrm{d}s}{s+\mathscr{C}}=\tfrac{1}{2}s^2-\mathscr{C}s+\mathscr{C}^2\ln(s+\mathscr{C})\\
&\quad\vdots\\
&\int\frac{s^n\,\mathrm{d}s}{s+\mathscr{C}}=\left(\sum_{i=0}^{n-1}\frac{(-\mathscr{C})^i s^{n-i}}{n-i}\right)+(-\mathscr{C})^n\ln(s+\mathscr{C})
\end{aligned} \tag{29}$$

As an example, a cubic polynomial approximation function on CVBE Γ_j is defined by

$$N_k(s)=(a_0+a_1s+a_2s^2+a_3s^3)_k \tag{30}$$

Then for $z_0\notin\Gamma_j$, expansion by partial fractions gives

$$\begin{aligned}
\int_{s=0}^{1}\frac{N_k(s)\omega_k\,\mathrm{d}s}{s+\mathscr{C}}=&\left[(a_3)_k\int_0^1 s^2\,\mathrm{d}s+(a_2-a_3\mathscr{C})_k\int_0^1 s\,\mathrm{d}s\right.\\
&\left.+(a_1-a_2\mathscr{C}+a_3\mathscr{C}^2)_k\int_0^1\mathrm{d}s+(a_0-a_1\mathscr{C}+a_2\mathscr{C}^2-a_3\mathscr{C}^3)_k\int_0^1\frac{\mathrm{d}s}{s+\mathscr{C}}\right]\omega_k
\end{aligned} \tag{31}$$

For $z_0\in\Gamma_j$, the Cauchy principal value is used as z_0 approaches the appropriate $z_i\in\Gamma_j$ (see the example of equations (20)–(25)).

FLUX-TYPE BOUNDARY CONDITIONS ON Γ

Normal flux boundary conditions of the potential function ϕ can be specified on an element Γ_j by using the Cauchy–Riemann relations with respect to tangential and normal co-ordinates along the straight-line segment CVBE Γ_j,

$$\frac{\partial \phi}{\partial s} = \frac{\partial \psi}{\partial n} \tag{32}$$

$$\frac{\partial \phi}{\partial n} = -\frac{\partial \psi}{\partial s} \tag{33}$$

Assuming that at least one value of the stream function ψ is specified on Γ as a reference, the remaining nodal ψ values can be equated by means of equation (33). The resulting linear approximation equation is simply an implicit finite difference relation between neighbouring nodal ψ-values, which is solved as part of the global matrix system.

MODEL SOLUTION

A global matrix system is developed composed of N equations for N unknowns. Each equation is either generated by a Cauchy integral approximation statement of equation (19) or by flux-type boundary conditions determined by equation (33). For a Dirichlet problem, or problems where the stream function is specified rather than using Neumann boundary conditions, a fully populated $N \times N$ matrix is generated for a N-variable system.

Solution of the global matrix system determines values for the unknown nodal point real variable functions on Γ. Interior values of the ϕ and ψ functions are evaluated by the Cauchy boundary integral for $z_0 \in \Omega$.

A comparison of the CVBE method to the real variable BEM[6,10] indicates that the CVBE method eliminates the need to numerically integrate boundary element contributions as is required in the BEM. Additionally, the CVBE method allows variable element trial function specifications on Γ without significant computer programming difficulty. However, the CVBE method does require some additional computational effort due to complex variable arithmetic.

EXAMPLE APPLICATIONS

Three example problems are presented to illustrate the use of CVBE methods.

Linear trial function on Γ_j

Hunt and Isaacs[14] developed a BIEM model based on the Cauchy integral theorem, but assumed that the complex function $\omega(z)$ was linear between boundary nodal points. The appropriate CVBE trial function is

$$\omega(z) = \omega_i\left(\frac{z_{i+1} - z}{z_{i+1} - z_i}\right) + \omega_{i+1}\left(\frac{z - z_i}{z_{i+1} - z_i}\right) \tag{34}$$

An extension of the model was then used by Hromadka and Guymon[15] to analyse moving boundary problems in a freezing/thawing two-dimensional soil problem.

The Hunt and Isaac's model is a particular subset of the CVBE method when $\phi(s)$ and $\psi(s)$ is specified as linear on each element, Γ_j. Thus,

$$\left.\begin{aligned} N_1(s) &= 1-s \\ N_2(s) &= s \end{aligned}\right\} \qquad 0 \leq s \leq 1 \tag{35}$$

and

$$\omega(s) = N_1\omega_1 + N_2\omega_2 \tag{36}$$

From equations (19) and (31), for $z_0 \neq \Gamma_j$

$$\int_{s=0}^{1} \frac{(N_1\omega_1 + N_2\omega_2)\,\mathrm{d}s}{s+\mathscr{C}} = \omega_1 \int_0^1 \frac{\mathrm{d}s}{s+\mathscr{C}} + (\omega_2 - \omega_1)\int_0^1 \frac{(s+\mathscr{C})\,\mathrm{d}s}{s+\mathscr{C}} - (\omega_2-\omega_1)\mathscr{C}\int_0^1 \frac{\mathrm{d}s}{s+\mathscr{C}} \tag{37}$$

where $\mathscr{C} = (z_1 - z_0)/(z_2 - z_1)$ and $(1+\mathscr{C})/\mathscr{C} = (z_2 - z_0)/(z_1 - z_0)$. Solving equation (37) gives

$$\int_{s=0}^{1} \frac{\omega(s)\,\mathrm{d}s}{s+\mathscr{C}} \equiv \omega_2\left[1 + \frac{(z_0 - z_1)}{(z_2 - z_1)} \ln\left(\frac{z_2 - z_0}{z_1 - z_0}\right)\right] - \omega_1\left[1 + \frac{(z_0 - z_2)}{(z_2 - z_1)} \ln\left(\frac{z_2 - z_0}{z_1 - z_0}\right)\right] \tag{38}$$

which determines the element contribution with $\omega(s)$ assumed linear on each Γ_j. In the case of $z_0 \in \Gamma_j$, the Cauchy principal value, equation (25), is used to determine the appropriate element contributions.

Several applications of the linear model are given in Hunt and Isaacs[14] and Hromadka and Guymon.[15] In these model applications, good results were obtained in the prediction of state variable values along the problem boundary, Γ, and in the interior of the domain, Ω. Hromadka and Guymon[15] used a linear trial function model to calculate heat flux values along the freezing front in a two-dimensional freezing soil problem. Based on estimates of net heat efflux, the changes in the freezing front co-ordinates are calculated assuming isothermal phase change of the available soil moisture. Re-evaluation of the Cauchy integral determines new values of heat efflux along the freezing front which are in turn used to relocate the moving boundary. Comparison of computed results to a finite element domain solution indicated that comparable modelling accuracy is achieved at a significant savings in computational effort.

Circular contour integration

Analogous to constant BEM elements, a simple approximation for $\omega(z)$ is to assume nodal values $\omega(z_i)$ are constant from mid-element to mid-element (Figure 8). This approximation results in the CVBE contributions of

$$2\pi i\omega(z_0) = \sum_{j=1}^{m} \left(\omega_j \int_{s=1/2}^{1} \frac{\mathrm{d}s}{s+\mathscr{C}_{j-1}} + \omega_j \int_{s=0}^{1/2} \frac{\mathrm{d}s}{s+\mathscr{C}_j}\right) \tag{39}$$

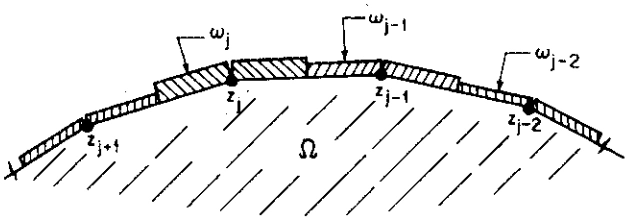

Figure 8. Constant boundary element approximation on global boundary Γ

where $z_0 \in \Omega$; $\mathscr{C}_{j-1} = (z_{j-1} - z_0)/(z_j - z_{j-1})$; $\mathscr{C}_j = (z_j - z_0)/(z_{j+1} - z_j)$. In equation (39), the global contour Γ is assumed subdivided by m nodal points. From equation (39), nodal value $\omega(z_j)$ is attributed the weighting

$$\omega_j\left(\int_{1/2}^{1} \frac{\mathrm{d}s}{s+\mathscr{C}_{j-1}} + \int_{0}^{1/2} \frac{\mathrm{d}s}{s+\mathscr{C}_j}\right) = \omega_j \ln\left[\frac{1/2(z_{j+1}+z_j)-z_0}{1/2(z_j+z_{j-1})-z_0}\right] \tag{40}$$

Equation (40) coupled with a Cauchy principal value term develops a general BIEM model, which has no variation of $\omega(z)$ in a neighbourhood of boundary nodal points.

This simple model can be used to demonstrate the convergence of the algorithm as the number of nodal points increases. The problem to be examined is the Cauchy integral along a circular contour with centre point z_0. In this problem, the number of nodes (m) is assumed to divide Γ into m equal length boundary elements. From equation (40) and Figure 9,

$$\omega_j \ln\left[\frac{1/2(z_{j+1}+z_j)-z_0}{1/2(z_j+z_{j-1})-z_0}\right] = \omega_j \ln\left[\frac{R_2\,\mathrm{e}^{i\theta_2}}{R_1\,\mathrm{e}^{i\theta_1}}\right] \tag{41}$$

where $R_2 = R_1$; and $\delta\theta \equiv \theta_2 - \theta_1$.

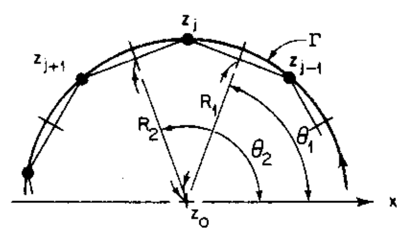

Figure 9. Definition of Cauchy integral on a circular contour

Therefore from equations (39) and (41),

$$2\pi i\omega(z_0) = \lim_{m\to\infty} \sum_{j=1}^{m} i\omega_j\,\delta\theta \tag{42}$$

Thus from the above, the Gauss mean value theorem results by

$$\omega(z_0) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} \omega(\theta)\,\mathrm{d}\theta \tag{43}$$

Although computer models based on the constant boundary element approach are simpler to formulate than higher order trial function approximations, the same magnitude of the matrix system results for a N nodal point problem as would occur when using a higher order trial function.

Higher order polynomial approximations

The complex variable analytic function $\omega(z)$ solves the governing Laplace equation in Ω and the various boundary conditions on Γ. The CVBE approach approximates $\omega(z)$ by the integration of complex variable approximation functions along the straight-line segments used to discretize the global boundary, Γ.

For $\omega(z)$ to be analytic in Ω and on Γ, $\omega(z)$ must not have poles or essential singularities in the problem domain. An important family of such problem solutions is the family of complex polynomials centred at the origin, $(0+0i)$, where the problem domain is linearly translated about the origin.

Consider a nth order complex variable polynomial solution to a Laplace problem. Then for the CVBE method to provide the exact solution, an nth order polynomial approximator $\hat{\omega}_n(s)$ must be specified on each straight line Γ_j by

$$\hat{\omega}_n(s) = \sum_{k=1}^{n} N_k(s)\phi_k + i \sum_{k=1}^{n} N_k(s)\psi_k \tag{44}$$

where on Γ_j

$$\begin{aligned} z(s) &= z_1 + (z_n - z_1)s, \qquad 0 \leq s \leq 1 \\ \hat{\omega}_n(z(s)) &= C_0 + C_1 z(s) + \ldots + C_n[z(s)]^n, \qquad 0 \leq s \leq 1 \end{aligned} \tag{45}$$

and the $N_k(s)$ are nth order polynomials in s. Then from the Cauchy integral theorem, each value $z_0 \in \Omega$ can be expressed as the sum of CVBE contributions

$$2\pi i \omega_n(z_0) = \sum_{\Gamma_j} \int_{s=0}^{1} \frac{\hat{\omega}_n(s)\,\mathrm{d}s}{s + \mathcal{C}_j}, \qquad \mathcal{C}_j = \left(\frac{z_1 - z_0}{z_n - z_1}\right)_j \tag{46}$$

But by assumption, $\hat{\omega}_n(s) = \omega_n(z)$ on each Γ_j. Therefore equation (46) can be rewritten as a strict equality by

$$\omega_n(z_0) = C_0 + C_1 z_0 + \ldots + C_n z_0^n \tag{47}$$

The polynomial model of equation (47) is simple to apply when using the polar co-ordinate relations

$$z_0 = R_0\, \mathrm{e}^{i\theta_0}; \qquad R_0 \geq 0,\ 0 \leq \theta_0 < 2\pi \tag{48}$$

$$z_0^n = R_0^n\, \mathrm{e}^{in\theta_0}, \qquad n \geq 0 \tag{49}$$

$$\mathrm{e}^{in\theta_0} = \cos(n\theta_0) + i \sin(n\theta_0) \tag{50}$$

In equation (47), the problem is to determine $2(n+1)$ unknown real value parameters for an order n complex polynomial by noting that

$$\omega_n(z) = (\alpha_0 + i\beta_0) + (\alpha_1 + i\beta_1)z + \ldots + (\alpha_n + i\beta_n)z^n \tag{51}$$

where the (α_j, β_j) are unknown real values to be determined based on the specified nodal boundary conditions. It should be noted that the CVBE approach simplifies to the nth order complex polynomial model only when (1) nth-order polynomial approximators are used on each straight-line complex boundary element; and (2) the nth order complex polynomial is the actual solution to the problem being studied. A major advantage offered by a complex polynomial model is the significant reduction in computational effort involved in the generation of global system matrices.

Applications of a complex variable polynomial model in the solution of steady-state and time-stepped steady-state field problems are contained in Hromadka and Guymon.[18] In that paper, the complex polynomial model was shown to produce more accurate results than a linear trial function model, but at a considerable saving in computational effort. For time-stepped problems, such as may be applied to slow moving interface problems, the complex polynomial model was found to reduce computational effort by over one-third of that required for the linear trial function model. However, the polynomial was found to be limited in the number of nodal points which could be used along the problem boundary without requiring higher precision computer arithmetic capability.

CONCLUSIONS

A generalization of real variable calculus-based BIEM and BEM techniques to the complex variable system is made. The new modelling approach allows for the combination of dissimilar trial functions to be defined along the problem boundary and eliminates the need for boundary element quadrature computations due to the direct integration of the complex variable trial functions. Using the derived integration formulae, the CVBE method is straightforward to prepare efficient computer programs. In this paper, direct comparisons are made to the linear trial function complex variable BIEM of Hunt and Isaacs,[14] the constant element BEM model of Brebbia,[6] and the complex variable polynomial approximation method.[18]

Applications have been made to problems where it is assumed Laplace's equation applies. The solution region must be homogeneous and isotropic. Theoretically, heterogeneous or anisotropic domains can be suitably transformed to homogeneous isotropic regions. Laplace's equation may be assumed from dynamic parabolic processes provided the temporal term may be assumed negligible. In this case approximate time-stepped solutions may be suitable. A large number of porous media fluid flow and heat flow problems may be approximated by these assumptions.

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