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APPLICATION OF A BOUNDARY INTEGRAL EQUATION TO PREDICTION OF FREEZING FRONTS IN SOIL

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ABSTRACT

A boundary integral equation formulation based on the complex Cauchy integral theorem is applied to two-dimensional soil-water phase change problems encountered in algid soils. The model assumes that potential theory applies in the estimation of heat flux along a freezing front of differential thickness and that quasi-steady-state temperatures occur along the problem domain boundary. Application of the boundary integral formulation to two-dimensional problems results in predicted locations of the freezing front which are highly accurate. Although the proposed formulation is based on the Cauchy integral theorem, similar models may be developed based on other forms of integration equation methods.

INTRODUCTION

The purpose of this paper is to apply well known potential theory to the solution of soil-water freezing or thawing problems wherein the temperature is a harmonic function. For problems where phase change latent heat effects dominate the heat transport process, the heat balance equations may be approximated by the Laplace equations coupled with the boundary conditions modified to include the effects of phase change. To do this, the dynamic component of the classical heat transport equation is assumed negligible compared to the latent heat term when freezing or thawing a soil-region. Moreover, it is necessary to assume an isotophic, homogeneous solution domain. However, by means of a suitable coordinate transformation for relatively geometrically simple regions, anisotropic or even heterogeneous domains may be transformed into a region in which potential theory may apply. As a result, boundary integral equation techniques may be applied which, for the test problems considered, significantly reduce computer storage and execution times when compared to classical domain methods. Generally, in freezing problems we are interested primarily in the location of the freezing front and in the estimation of heat flux values normal to the freezing front. Boundary integral equation methods (B.I.E.M.) focus on these two types of problems directly. Additionally, the B.I.E.M.-based models can significantly reduce the computational effort involved in producing mesh generations and manipulations, besides allowing very small mesh relocation except along a moving feeezing-front boundary. In this paper, we will develop a model based on a sophistication of a boundary integral equation method utilizing Cauchy's integral theorem for analytic functions as presented by Hunt and Isaacs (1981), and will show results that support its use in geothermal problems involving freezing or thawing that are associated with geotechnical problems typical to areas where there is significant penetration of the freezing isotherm.

The application of potential theory is illustrated by considering a heat flow problem defined on a connected domain Ω with an exterior boundary described by a simple closed contour Γ. The heat flux across a surface located in the interior of Ω is given by

$$\vec{Q} = -K\vec{\nabla}\phi \tag{1}$$

where ϕ is the temperature potential, and K is the thermal conductivity which is assumed isotropic and constant in Ω.

For two-dimensional problems the heat flux may be represented by complex variable notation as

$$Q = Q_x + iQ_y = -K\frac{\partial \phi}{\partial x} - iK\frac{\partial \phi}{\partial y} \tag{2}$$

where $i = \sqrt{-1}$.
For an arbitrary simple closed contour C, interior of Ω

$$\int_C Q_\mathrm{n}\, \mathrm{d}s = \int_C Q_\mathrm{t}\, \mathrm{d}s = 0 \tag{3}$$

where Q_n and Q_t are outward normal and tangential components of the heat flux on C; steady state conditions are assumed in Ω with no thermal sources or sinks. Equation (3) defines ϕ as harmonic, satisfying the Laplace equation in Ω. The harmonic conjugate ψ exists in Ω and is related to ϕ by the Cauchy–Riemann equations (Dettman, 1969). The complex temperature $\xi(z)$ is defined by

$$\xi(z) = \phi(x,y) + i\psi(x,y) \tag{4}$$

where $\psi(x,y) = k_0$ represents constant heat flux lines and $\phi(x,y) = k_0$ represents an isothermal potential where ϕ is constant along each potential.

Classes of problems which can be described in terms of a complex function such as (4) include ideal fluid flow, heat flow, electrostatics and porous media flow in isotropic, homogeneous domains. For approximate solutions, domain numerical approximations are generally employed such as the finite element, finite difference, or nodal domain integration methods. For homogeneous problems, boundary integral equation methods are reported to be superior to the domain numerical methods (Liggett, 1977; Liggett and Liu, 1977; Lennon et al., 1980).

In this paper, a boundary integral method is used to solve for the temperature distribution on the problem boundary Γ where two types of boundary condition are approximated simultaneously on $\Gamma_1 + \Gamma_2 = \Gamma$:

$$\begin{aligned} \phi(x,y) &= \phi_0(x,y), \quad (x,y) \in \Gamma_1 \\ \psi(x,y) &= \psi_0(x,y), \quad (x,y) \in \Gamma_2 \end{aligned} \tag{5}$$

and isothermal phase change is defined on $\Gamma_3\,\Gamma_1$ such that

$$L\frac{\mathrm{d}s}{\mathrm{d}t} = \sum_i Q_{\mathrm{n}_i} \tag{6}$$

where the subscript i is a summation index, and where s and Q_{n_i} are normal spatial and heat flux terms, respectively.

Equation 6 indicates that the rate of boundary movement (freezing front) due to an isothermal phase change is directly related to the net heat efflux on the boundary of Γ_3. Additionally, the sum of normal heat flux terms along Γ_3 is defined in terms of algebraic sign according to the freezing/thawing direction. In eqn. (6), the volumetric latent heat of fusion for soil-water is used for L where the class of problems to be studied is the prediction of the freezing-front location within a homogeneous isotropic (or regional homogeneous isotropic) freezing/thawing soil where soil-water transport is assumed negligible. From eqn. (6), the freezing front contour Γ_3 is redefined in (x,y) coordinates after each time advancement. An approach used to calculate new Γ_3 (x,y) coordinates is to displace the nodal points (located at the midpoint of each boundary element) normal to the heat flux direction such that the total Γ_3 movement balances the total heat efflux integrated with respect to time along Γ_3. The normal heat flux values are calculated along Γ_3 by the Cauchy–Riemann relations for analytic (complex variable) functions. For the examples considered, results indicate that the nodal x coordinates vary considerably less than the y coordinates and that carefully defined subregions of two-dimensional problems may be compared to one-dimensional model results to check modeling accuracy.

The approach used to develop a two-dimensional soil-water freezing/thawing model is to approximate a two-dimensional dynamic temperature field with a time-stepped steady state temperature distribution in Ω by means of a complex boundary integral formulation. In soil-water phase change problems where latent heat effects dominate the transient heat evolution, a quasi steady-state type problem can be formulated for many real world situations where a steady state head flux estimation is a good approximation for the time-averaged dynamic heat flux values.

Such an approach includes the advantage of a precisely defined freezing front location in a two-dimensional domain Ω without the use of the finite element deforming-grid method (such as used in Lynch and O'Neill, 1981) or a multi-dimensional finite element model. Consequently, the proposed model may be ideal for many homogeneous soil freezing/thawing problems in geotechnical engineering where computer capability is limited, such as can be obtained with present day microcomputers.

FORMULATION

A discussion of the current trends in cold-regions thermal numerical models is contained in Guymon et al. (1980) and a review of current boundary integral methodologies is contained in Hunt and Isaacs (1981). The general trend in algid-soil numerical models is to use domain methods to approximate the dynamic heat flow equation and to include a soil-water phase change model incorporating an isothermal or apparent heat capacity approximation. These domain numerical models allow the ease of solving for problems which have a spectrum of dissimilar materials and anisotropy. However for problems which are homogeneous, a B.I.E.M. formulation may be used to estimate the heat flux values within the problem, resulting in a numerical model which, for the test problems considered, is a fraction of the size of the domain model. However, because the B.I.E.M. model develops a fully populated Nth order matrix for an N nodal point discretization, the global model size can quickly exceed a domain model's requirements when a fine mesh is used on the problem boundary.

THEORY

Figure 1 shows an assumed problem to be studied which consists of a roadway embankment. A constant temperature is specified for ϕ_U and ϕ_L with the sides of the roadway embankment problem being specified with values of $\psi = \psi_L$ and ψ_R. For a boundary integral formulation, Neumann boundary conditions can be used on the left and right sides rather than determining ψ_L and ψ_R or an equivalent ϕ_L and ϕ_R. Any of the usual boundary integral approaches can be used for this problem; a complex formulation is used in this study due to the ease in contour integration which results from the well known Cauchy's integral theorem.

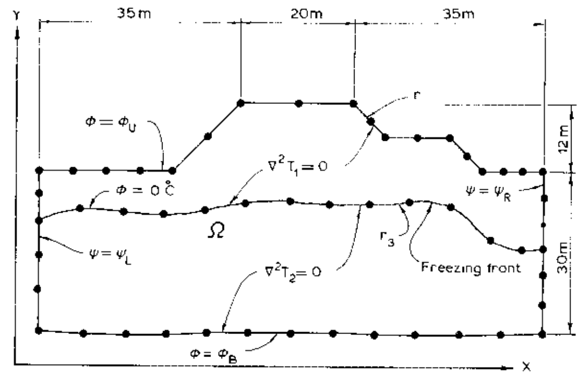

Fig. 1. Roadway embankment problem showing B.I.E.M. nodal placement and freezing-front definition.

Assuming the freezing-front location to be defined at some time t_0, the dynamic heat evolution problem is approximated by solving the Laplace relations (Fig. 1) to estimate the mean heat flux values along the freezing-front location during a large timestep, Δt. For example, in the problems studied, time steps of one week are used with good results. From the estimated heat flux values, the change in the freezing front is calculated from eqn. (6). That is, a method of calculating the change in the freezing-front coordinates is to calculate the change in the nodal point coordinates in the direction of net normal heat flux. For nodal points located at the midpoint of boundary elements (Brebbia, 1978), the determination of new coordinates at the freezing front may be estimated by a simple balance between the volume of soil frozen and the time-integrated heat evolved. Due to the model's basic assumption of phase change effects dominating the entire heat transport process, the freezing-front evolution is relatively slow; the simple freezing-front evolution model described above was found to be adequate for the problems tested. The new freezing front location at time $t_0 + \Delta t$ is then used to obtain a better estimate of average heat evolution balance using heat flux approximations for time $(t_0 + \Delta t/2)$.

To solve the Laplace equations, $\xi(z)$ is solved on the boundary of each subproblem to determine simultaneous heat flux values at each nodal point along the freezing front. From Cauchy's theorem,

$$\xi(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{\xi(z)\,dz}{(z - z_0)} \tag{7}$$

where z $(z = x + iy) \in \Gamma$, and z_0 is in the interior of Ω. Integration is performed in the usual counterclockwise fashion on boundary Γ.

by definition of the contour integral

$$\int \frac{\xi(z)\,\mathrm{d}z}{(z - z_0)} = \int_{\Sigma \Gamma_{j,j+1}} \frac{\xi(z)\,\mathrm{d}z}{(z - z_0)} \tag{8}$$

where $\Gamma_{j,j+1}$ is the contour between nodes z_j and z_{j+1}. For z_j on Γ,

$$\xi(z_j) = \frac{P}{\theta i} \int_{\Gamma} \frac{\xi(z)\,\mathrm{d}z}{(z - z_j)} \tag{9}$$

where P is the Cauchy Principal Value, and θ is the interior angle between line segments $\Gamma_{j,j+1}$ (Fig. 2). On the boundary Γ of each subproblem, either $\psi(z_j)$ or $\phi(z_j)$ is defined at each boundary nodal point $z_j \in \Gamma$. Consequently, from eqn. (9), N equations result for the N unknown variables for an N-nodal point discretization of Γ. The integrations on Γ are approximated by means of combining eqns. (8) and (9).

The numerical integrations from solving eqns. (8) and (9) are arrived at by assuming $\xi(z)$ to be piecewise linear between nodal points, that is

$$\hat{\xi}(z) \equiv \left(\frac{z - z_j}{z_{j+1} - z_j}\right)\xi_{j+1} + \left(\frac{z_{j+1} - z}{z_{j+1} - z_j}\right)\xi_j \tag{10}$$

where $\hat{\xi}(z)$ is the approximation of $\xi(z)$ on $\Gamma_{j,j+1}$; and ξ_j is a nodal point value at node z_j.

The numerical integrations result in a complex logarithm formulation which is given in Hunt and Isaacs (1981) and is summarized by the following:

$$P \int_{z_{j-1}}^{z_{j+1}} \frac{\xi(z)}{(z - z_j)}\,\mathrm{d}z = \xi_{j+1} - \xi_{j-1} + \xi_j \ln\left|\frac{z_{j+1} - z_j}{z_{j-1} + z_j}\right| \tag{11}$$

for $z_j \in \Gamma$. For boundary nodal points $z_0 \neq j,j+1$

$$\int_{\Gamma_{j,j+1}} \frac{\xi(z)\,\mathrm{d}z}{(z - z_0)} = \xi_{j+1}\left[1 + \left(\frac{z_0 - z_j}{z_{j+1} - z_j}\right)\ln\left(\frac{z_{j+1} - z_0}{z_j - z_0}\right)\right] - \xi_j\left[1 + \left(\frac{z_0 - z_{j+1}}{z_{j+1} - z_j}\right)\ln\left(\frac{z_{j+1} - z_0}{z_j - z_0}\right)\right] \tag{12}$$

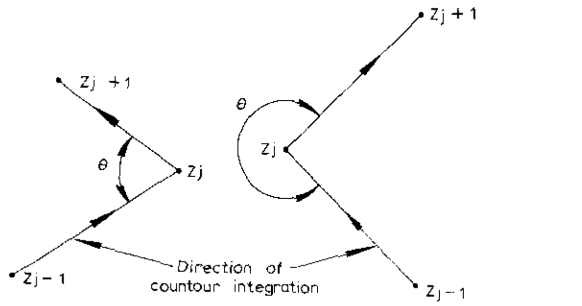

Fig. 2. Geometric definition of complex contour angle, θ.

The contour integrations result in the equation system

$$\xi_j = \frac{\xi_{j+1} - \xi_{j-1} + \sum_{k \neq j,j+1} \int_{\Gamma_{k,k+1}} \frac{\xi(z)\,\mathrm{d}z}{(z - z_j)}}{\ln\left|\frac{z_{j-1} - z}{z_{j+1} - z_j}\right| + i\theta} \tag{13}$$

The full system is written in matrix form

$$\underset{\sim}{K}(\underset{\sim}{\phi}, \underset{\sim}{\psi}) = \underset{\sim}{0} \tag{14}$$

where $\underset{\sim}{K}$ is a fully populated matrix of known coefficients from eqn. (13) and $(\underset{\sim}{\phi}, \underset{\sim}{\psi})$ is the array of (ϕ_j, ψ_j) values of the complex temperature ξ_j. Heat flux values can be estimated by determining a ξ_0 interior of Ω and calculating the normal temperature gradient values using each $\phi_0(\phi_0 = 0°C)$ value along the freezing front, Γ_3, or directly from the Cauchy–Riemann equations.

Since the problem domain is assumed homogeneous and isotropic, parameter estimations are not required on a control volume basis because the soil mixture is assumed either entirely frozen (with some predetermined unfrozen water content typical of these problem freezing temperatures) or thawed on either size of the freezing front, Γ_3. In comparison to a finite-element fixed-grid model, thermal parameters are constantly changing as the control volume ice content values change with time, necessarily causing frequent domain-method global matrix regenerations due to the linearized approximations of the governing nonlinear heat flow equation.

For an anisotropic homogeneous problem, the above methodology can be utilized by simply rescaling the global problem to accommodate the ratio of horizontal-to-vertical thermal conductivity values

(Myers, 1971) and solving the modified problem in the new $(\hat{x},\hat{y})$ space.

For nonhomogeneous problems, complexities arise in an effort to simultaneously solve for the unknowns of ξ_j shared on the boundaries of homogeneous regions. The resulting extra manipulations often overshadow the benefits of the proposed geothermal model, especially when comparing the effort required for problem data-input preparation between a domain method formulation and the boundary integral formulation. Additionally, in the approximation of heat evolution in a nonhomogeneous problem, several nodal points are required in the interior of Ω along the boundaries of the defined homogeneous regions, resulting in a significant increase in computational effort due to the fully populated matrix requirements of a boundary integral formulation.

For computational efficiency and model comparison purposes, a B.I.E.M. numerical model (Brebbia, 1978) was also used for the approximation of nodal temperatures and heat flux values on the freezing-front boundary. Both the complex temperature formulation and the "boundary element method" were applied to identical nodal placements and definitions for each test problem studied. Comparison of computational results indicate that either approach produces similar values of heat flux on the freezing-front boundary and, therefore, predict nearly identical freezing-front locations. An advantage of the complex temperature formulation is the relatively straightforward contour integrations summed along the boundary Γ. An advantage of the boundary element formulation is the direct computation of heat flux values along the freezing-front nodal points.

APPLICATIONS

The boundary integral equation method (B.I.E.M.) geothermal model was applied to the roadway embankment problem schematically defined in Fig. 1. The first problem studied assumed that a constant $-10°C$ temperature is uniformly imposed along the top surface of the domain with the rest of the domain initially set at $+0.1°C$; that is,

$$\phi_U = -10°C$$

$$\phi_B = +0.1°C$$

(ϕ_R,ϕ_L) = Linear variation from (ϕ_U,ϕ_B) to Γ_3

For the subproblem $\nabla^2 T_1 = 0$, a frozen thermal conductivity value of ice is used whereas an unfrozen thermal conductivity of water is used for subproblem $\nabla^2 T_2 = 0$. The freezing front Γ_3 is initially assumed at a location of 50 cm below the top ground surface. Figure 3 shows several plots of predicted freezing-front locations from the geothermal model.

To test the accuracy of the model, three quasi

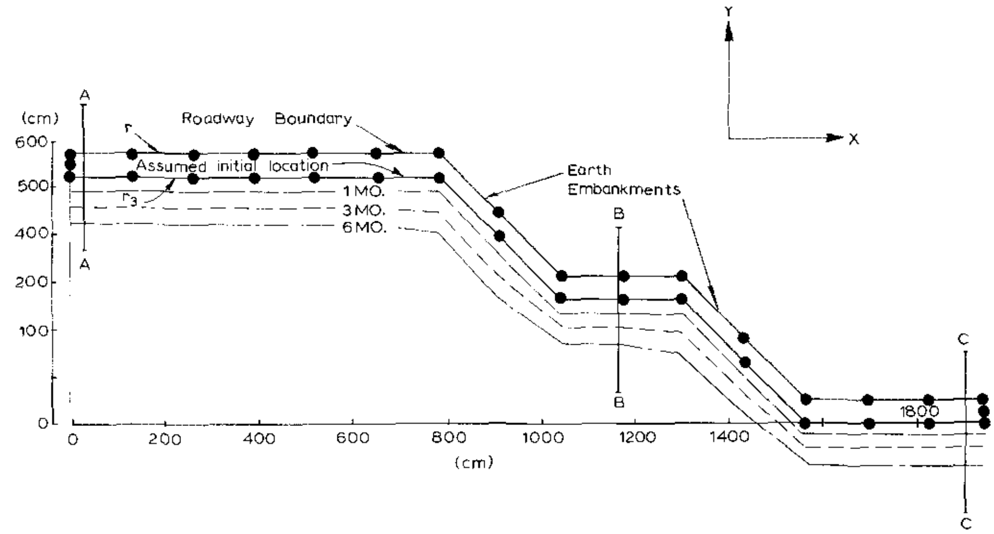

Fig. 3. Freezing-front locations as estimated by geothermal model (34 nodal-point B.I.E.M. model).

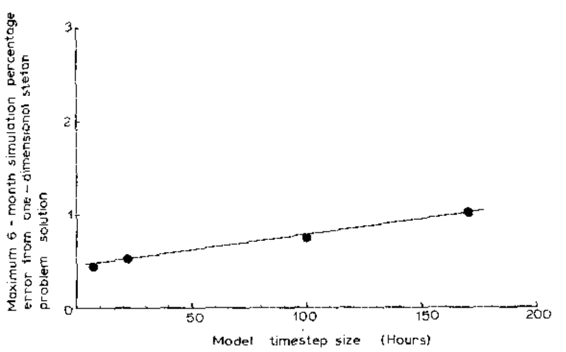

Fig. 4. Sensitivity of B.I.E.M. geothermal model to time-step size.

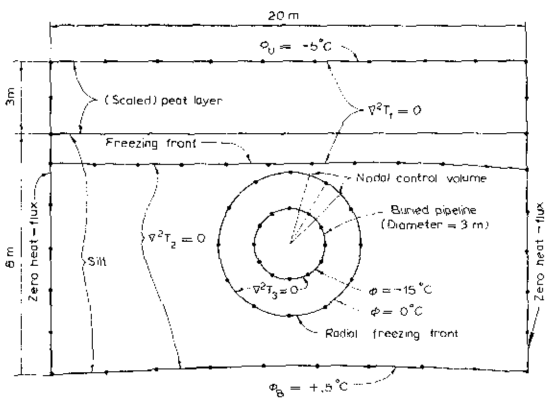

Fig. 5. Buried-pipeline problem showing two freezing-front definitions and three temperature-domain solutions.

one-dimensional problems were identified (labeled A-A, B-B, C-C) in the two-dimensional problem domain of Fig. 3, and the one-dimensional analytical solutions compared to the two-dimensional geothermal model results. The geothermal model was found to give good results (Fig. 4) when compared to the well known one-dimensional semi-infinite Stefan problem solution.

To examine the effects of model time-step size, the geothermal model was tested for variations in the prediction of the freezing-front location as a function of the length of the time-step size. The variation in results are shown in Fig. 4 which indicates that the geothermal model results vary in accuracy by less than 2% between time-step sizes of 6 hours and 1 week.

Due to the small computer memory requirements for matrix solution and the relatively simple coding necessary for the B.I.E.M. or boundary element (B.E.M.) approaches, the freezing-front locations plotted in Fig. 3 were obtained from a well known 64 K byte microcomputer capability (FORTRAN language). Consequently, sophisticated two-dimensional problems may be analyzed economically by currently available low-cost "household" digital computers.

As a second example problem, the effects of a buried pipeline maintained at subfreezing temperatures were studied. In this problem the freezing-front model used in the first example problem was utilized along with a radially defined freezing front from the buried pipeline (Fig. 5).

Three subproblem domains were defined in order to estimate a net efflux of heat along each freezing front. To verify the model approximations, one-dimensional problem solutions were compared to both the horizontal-plane freezing front and the radial freezing-front problems. The problem simulation time was for a 6-month duration and required less than 10 CPU seconds on a Data General Eclipse mini-computer system (using a model time-step size of one week). Model accuracy was found to be in error of less than 2% from the one-dimensional Stefan problem analytical solutions. On a 64K byte (FORTRAN) microcomputer system, computer requirements cost approximately 2 CPU minutes for the same 6-month simulation. From Fig. 5, it can be noted that a peat layer is approximated by rescaling the global model and neglecting horizontal heat flux effects and assuming only a vertical freezing-front penetration in the peat (the modeled results verified this assumption). Heat flux was assumed to be zero on the left and right sides of the global problem domain. Computer solution of the steady-state heat flux problems were obtained by the B.I.E.M. (Brebbia) in order to directly obtain heat flux values along the radial freezing front.

CONCLUSIONS

A soil-water freezing geothermal model is developed which is based on a complex variable boundary integral equation method using Cauchy's theorem.

The geothermal model provides good results in the prediction of soil-water freezing fronts in two-dimensional problems. Since the latent heat effects of soil-water phase change are assumed to dominate the total heat evolution budget, quasi steady-state temperature distributions may be used to estimate total net heat flux values along the soil-water freezing fronts. The computer coding requirements are small, enabling the model to be accommodated on currently available home microcomputers for many two-dimensional freezing/thawing soil problems.

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