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A PROBABILISTIC-DETERMINISTIC ANALYSIS OF ONE-DIMENSIONAL ICE SEGREGATION IN A FREEZING SOIL COLUMN

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ABSTRACT

A deterministic model of frost heave based upon simultaneous analysis of coupled heat and moisture transport is cascaded with a probabilistic model of parameter variations. The multiparameter, deterministic model is based upon submodels of moisture transport, heat transport, and lumped isothermal freezing processes. The probabilistic model is based upon Rosenblueth's method which only requires knowledge of parameter means and their coefficients of variation. The deterministic model is relatively insensitive to thermal parameter variations because the phase change process dominates the thermal regime of a freezing moist soil. The model is sensitive to hydraulic parameters which control the rate mobile liquid water is drawn into the freezing soil region. Four hydraulic parameters were varied within reported and assumed levels of parameter variation for two soils; a frost-susceptible silt and a marginally frost-susceptible dirty gravel for which laboratory data on parameters and frost heave were available. The resulting frost heave variations were fit to a beta distribution and confidence limits of at least 95% were predicted within two sigma bounds. The coefficient of variation of unfrozen hydraulic conductivity primarily determines the coefficient of variation of simulated frost heave. Comparison of these results with two detailed field cases indicates a close comparison with beta distribution parameters.

INTRODUCTION

Attempts to develop mathematical models of frost heave have centered on continuum-deterministic approaches which are sometimes referred to as conceptual or physics based approaches. Guymon et al. (1980) and Hopke (1980) review these efforts. Generally, mathematical models have included simultaneous heat and moisture transport in a one-dimensional column. While there is almost total agreement that these two processes must somehow be included in any model of frost heave, there is considerable uncertainty and disagreement on the ice segregation processes itself. The processes occurring in the freezing zone are, unfortunately, poorly understood. There also are somewhat divergent objectives in developing these models as explicitly or implicitly viewed by the various authors, and these differences in objectives have led to differences in approaches. One purpose of this paper is to present a more systematic basis for viewing or judging the frost heave modeling exercise.

Guymon et al. (1981) have questioned the fundamental concept of using deterministic models because of uncertainty concerning model parameters and modeling concepts of the ice segregation process. They suggest that probabilistic concepts should be coupled with deterministic approaches.

Chamberlain (1980) has recently conducted comprehensive studies of field frost heave for a small section of roadway in Hanover, NH, with sandy silts as the base material. The variations of frost heave were carefully measured at 455 discrete points and were fitted to a beta-probability distribution, suggesting that frost heave can be evaluated as a probabilistic process. Chamberlain observed a coefficient of variation of about 72% and a maximum and minimum frost heave of about 3 and 2 standard deviations, respectively. The maximum mean frost heave he observed was 4.2 cm and frost penetration was 90 cm. Numerous "undisturbed" samples collected at this site for determining hydraulic parameters should be useful in defining the statistical properties of these parameters which may in turn be related to the observed frost heave variations. We have made similar measurements at 39 discrete points on a taxiway at the Albany County Airport in Albany, NY. A maximum mean heave of 1.65 cm was observed on 11 February 1980 when frost had penetrated less than 50 cm into the silty-sand sub-base material. A maximum 130% coefficient of variation for heave was obtained. Observed maximum and minimum heave values 3.35 cm and 0.9 cm are equivalent to approximately plus and minus two standard deviations. Cores of frozen ground where ice lensing has occurred also suggest a process exhibiting random features. Ice lenses will occur in a seemingly random pattern in a core only a few cm in diameter. Analysis of such a "one-dimensional" core uniformly frozen so that there would be a generally uniform distribution of ice, indicated little relation between the frequency of ice lenses along three vertical transects taken 4 cm apart. Ice lenses were on the order of 0.4 mm thick. An ice lens might occur on one horizontal transect and just 4 cm horizontally to the right or left no ice lens was visible.

A central objective of this paper is to develop a probabilistic concept of frost heave which includes the usual deterministic approaches but recognizes the inherent discrete nature of porous media. To do this, a new probabilistic approach will be presented which avoids the commonly used and sometimes expensive Monte Carlo method. Although a particular mathematical model will be used (Guymon et al., 1980) to develop a probabilistic-deterministic model, the theory developed here is generally applicable to deterministic models proposed by others.

SYSTEMS CONCEPTS

Figure 1 is an approach to viewing the modeling process. The prototype system S (e.g. a laboratory soil column) is subject to excitations, x (or inputs) which are spatially and temporarily distributed. Spatially and temporally distributed outputs are observed. Inputs or boundary conditions may be subfreezing temperatures, water table location, and surface surcharge (overburden). Outputs may be frost heave, y, or soil pore pressure, temperatures, or ice content. Because it is usually impossible to measure x exactly, subsystem X indicates a model process to determine an index, x', of x which has some error. In our case we are generally lumping x in space but are preserving, to the extent possible, any low-frequency dynamic characteristics of x. Since our deterministic model M is based upon the continuum assumption, certain parameters arise in the model derivation which purport to characterize S (e.g. thermal conductivity or hydraulic conductivity). Subsystem P indicates this modeling or sampling process which yields imperfectly known parameters, p_i. Model outputs, y', will therefore be imprecise but may be compared to imperfect observations of y for

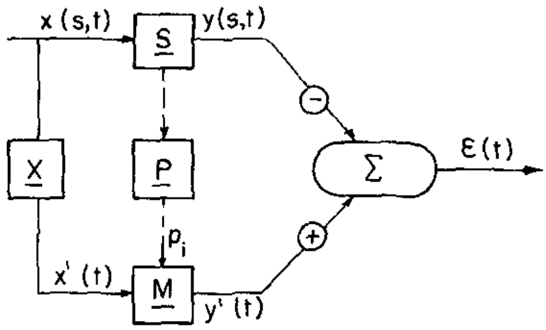

Fig. 1. A schematic of the modeling process showing modeling uncertainty.

some bounded time period to determine model uncertainty, $\epsilon(t)$, where

$$\epsilon(t) = y'(t) - y(t) \tag{1}$$

We are considering y as lumped in order to make this computation. Modeling uncertainty is arbitrarily grouped into four general areas:

1. Errors α_1 due to the choice of M which includes the choice of a numerical analog.
2. Errors α_2 due to spatial and temporal discretization and averaging.
3. Errors α_3 due to boundary conditions (i.e. choice of X) and due to choice of initial conditions.
4. Errors α_4 due to the selection of p_i; i.e. choice of P.

The total model uncertainty is some function of the α_i errors

$$\epsilon(t) = \epsilon(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \tag{2}$$

where the α_i errors may be interrelated and ϵ may be non-stationary. Because of approximations necessarily incorporated in the model there will obviously be some error or uncertainty in model predictions. Furthermore, the complex and nonlinear nature of the model requires comparison to prototype situations to evaluate its precision.

An investigation of errors associated with the choice of numerical methods (e.g. finite difference and finite element methods) and spatial and temporal discretization is the subject of a future paper. Boundary condition effects will also be investigated in a future paper.

The major thrust of this paper will concentrate on parameter errors associated with an assumed uniform soil profile. It is well known (Harr, 1977; Nielsen et al., 1973; Warrick and Nielson, 1980) that considerable variation is observed in field soils that are within a small geographical area and may be generally classed as the same soil. Moreover, most field soil profiles are layered (non-uniform) although they are often simplified to an "average" uniform soil profile for analysis. This usually unknown variability coupled with sampling and measuring errors is significant. Frost heave is an ideal process to study by probabilistic methods since it is sensitive to minor variations in soil properties and environmental conditions. Moreover, frost heave or ice segregation processes effectively integrate all hydraulic, thermal and chemical processes taking place in a finite column of soil so that by measuring one output, frost heave, all other processes are indirectly sampled. Frost penetration depths can also be accurately measured and they provide some indication of parameter variability effects.

BRIEF REVIEW OF DETERMINISTIC MODEL

Guymon et al. (1981) describe the deterministic model, and these details will not be repeated here. Only a general summary of the model will be presented.

The model is applicable to a saturated or unsaturated, one-dimensional, vertical, soil column which is subjected to time-dependent variations in upper and lower boundary temperature and pore water pressure conditions. The upper boundary condition is assumed to be a no-moisture flux condition if the soil surface is frozen. Additionally, a surface surcharge or overburden condition is accommodated by the model. Major assumptions employed in the model are:

1. Unsaturated moisture flow theory applies and Darcy's law is valid in the unfrozen zone and the freezing fringe.
2. Moisture movement is by liquid films driven by the total hydraulic head energy gradient.
3. Moisture movement in frozen zones is negligable.
4. The well known heat equation, including a sensible heat advection term, applies to the entire soil profile.
5. The unfrozen zone is nondeformable and the frozen zone is only deformable due to ice lens growth.
6. The fluid sink due to freezing and the latent heat process may be decoupled into an isothermal approach (i.e. a heat balance process).
7. Ice segregation occurs when moisture drawn into the freezing zone exceeds the soil porosity minus an unfrozen water content factor, corrected for volumetric ice expansion.
8. Hysteresis is not present and all functions are single valued and piece-wise continuous.
9. Overburden and surcharge effects are included

by adding these pressures to pore water pressures at ice segregation fronts only.

10. Salt transport effects are negligible; i.e. the freezing point depression of soil water is constant, and the unfrozen moisture content at a given temperature is constant.
11. Constant parameters are invariant with respect to time; i.e. they do not change in response to freeze–thaw cycles.

The model is based upon simultaneous solution of partial differential equations of heat and moisture flux in the unfrozen zone wherein it is assumed unsaturated flow is modeled by unsaturated flow theory

$$\frac{\partial [K_H(\partial\phi/\partial x)]}{\partial x} = \frac{\partial\theta_u}{\partial t} + \frac{\rho_w}{\rho_i}\frac{\partial\theta_i}{\partial t} \qquad (3)$$

and the well known sensible heat conduction–advection equation

$$\frac{\partial [K_T(\partial T/\partial x)]}{\partial x} - C_w v \frac{\partial T}{\partial x} = C_m \frac{\partial T}{\partial t} - L_a \frac{\rho_w}{\rho_i}\frac{\partial\theta_i}{\partial t} \qquad (4)$$

where

x = positive coordinate downward
t = time
ϕ = total head = $\psi - x$ (where ψ = pore pressure head)
θ_u = volumetric unfrozen water content
θ_i = volumetric ice content
ρ_w, ρ_i = density of water and ice respectively
K_H = $K(\psi)$ = hydraulic conductivity
T = temperature
K_T = thermal conductivity of soil–water–ice mixture
L_a = volumetric latent heat of fusion for bulk water
C_w = volumetric heat capacity of water
C_m = volumetric heat capacity of soil–water–ice mixture

The moisture sink and latent heat components of eqns. (3) and (4) are decoupled and are solved using an isothermal approximation (Hromadka et al., 1981). These components only exist in freezing or thawing zones of the soil profile.

Figure 2 illustrates the computation process at a specific time level. An overburden pressure (as head of water), ψ_o, is shown. Also the unfrozen water content factor, θ_n, and soil porosity, θ_o, are shown. The primary variables ψ and T are computed from eqns. (3) and (4) and the secondary variables θ_u, θ_i, and θ_s (segregated ice) are computed. From this latter quantity, lumped heave is computed as shown in Fig. 2.

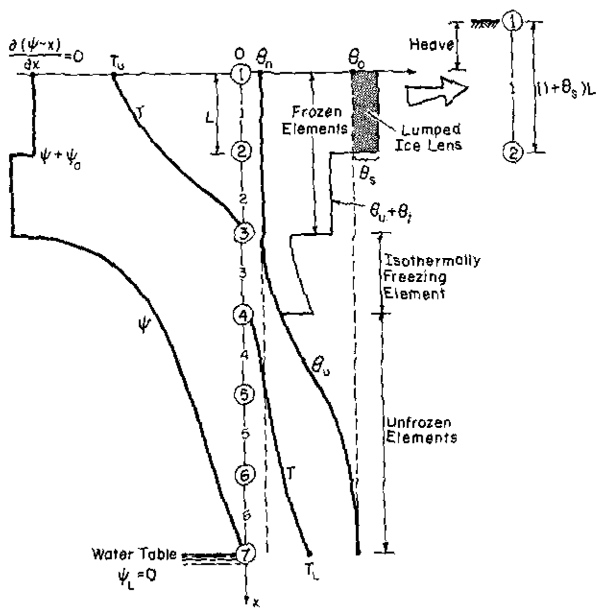

Fig. 2. Illustration of deterministic model results at a given time level (from Guymon et al., 1981).

TABLE 1

Soil parameters required for the deterministic model

Parameter	Description
n, A_w	Characterize soil water versus pore pressure relationship for unfrozen soil
$K(\psi)$	Unfrozen hydraulic conductivity versus pore pressure relationship
E	Parameter to correct $K(\psi)$ for ice in freezing zone (as function of θ_i, ice content)
θ_o	Porosity
θ_n	Unfrozen water content factor
C_s	Volumetric heat capacity of soil
K_s	Thermal conductivity of soil
ρ_s	Soil density
T_f	Freezing point depression of soil water

Numerical solution is by the nodal domain integration method (Hromadka and Guyman, 1981) which requires a spatial discretization of the order of 1 cm. The time domain solution is by the Crank–Nicolson method which requires temporal discretization on the order of 0.5 h and a parameter update frequency on the order of 1.0 h. Equations are temporarily decoupled during discrete time periods and parameters are assumed constant within each discrete spatial solution domain.

The model is a multiparameter model and is highly nonlinear. Table 1 lists the soil parameters required by the model. The heat and fluid transport processes are coupled through the latent heat process and the parameters that must be employed in the derivations of eqns. (3) and (4). Fortunately, soil systems are usually highly damped, permitting reasonable approximations without employing costly iteration techniques. The model is efficient and can be solved on the miniclass computers using FORTRAN IV language.

PROBABILITY MODEL

Freeze (1975) and others have investigated the combination of stochastic and deterministic models. In particular, Freeze considered the problem of groundwater flow in a non-uniform one-dimensional homogeneous medium. On the basis of his study, Freeze had "doubts about the presumed accuracy of the deterministic conceptual models that are so widely used in groundwater hydrology." If he had doubts about a similar but simpler system, we must confess considerable pessimism about deterministic models of the more complex ice segregation processes. Only a few parameters were of concern to Freeze (1975), but we are considering the 10 parameters shown in Table 1. Values of the heat capacity, thermal conductivity, density and latent heat capacity of water and ice were obtained from standard tables.

Freeze's (1975) stochastic analysis was based upon the well known Monte Carlo technique (Harr, 1977) which requires an assumption of the statistical distribution of the stochastic variables. Freeze assumed porosity was normally distributed and saturated hydraulic conductivity was log-normally distributed, and used 500 Monte Carlo runs for each parameter. Values were randomly generated from an assumed probability distribution and were applied to a deterministic model. Typically, most investigations of this nature use a large number of runs, i.e. 500 or more. Because of the apparent need for many Monte Carlo runs, this type of stochastic analysis can be expensive, particularly if the variance is non-stationary for the type of dynamic problems being considered and if the variance should be significantly different for different soil types.

An alternative approach to the Monte Carlo method is proposed that is based upon Rosenblueth's (1973) method, but because this method is not widely known in the open literature, a heuristic derivation of the method will be presented for the case of one random variable. Suppose

$$y = f(x) \tag{5}$$

where $f(x)$ is an arbitrary, continuously differentiable frequency distribution of the random variable x. The usual procedure for determining the expectance or mean and the second moment or variance of y is to expand y around $\bar{x}$ (the mean of x) using a Taylor's series (Harr, 1977).

Alternatively, one may use Rosenblueth's method which is analogous to determining the required simple support forces for a statically stable weightless beam with an arbitrary load (Fig. 3). Now suppose $f(x)$ has been normalized so that the total load equals unity, i.e.

$$1 = \int_0^L f(x)\mathrm{d}x \tag{6}$$

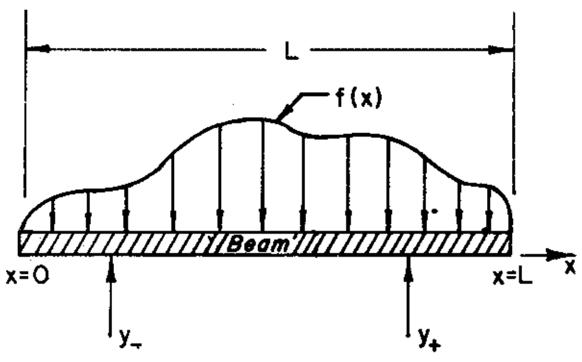

Fig. 3. Schematic illustrating Rosenblueth's method.

Also, suppose the mean value of x and its coefficient of variation are known for N discrete values of x, where these statistical properties are defined in the usual manner

$$E(x) = \bar{x} = \frac{1}{N}\sum_{0}^{N} x_i \qquad (7a)$$

$$V_x = E(x^2) - [E(x)]^2 = \frac{1}{N}\sum_{0}^{N}(x_i - \bar{x})^2 \qquad (7b)$$

where V_x is the variance of x. The coefficient of variation is given by

$$CV = S_x/\bar{x} \qquad (8)$$

where S_x is the standard deviation and is equal to the positive square root of the variance. Adopting the notation

$$\begin{aligned} y_+ &= f(\bar{x} + S_x) \\ y_- &= f(\bar{x} - S_x), \end{aligned} \qquad (9)$$

static equilibrium of the beam shown in Fig. 3 will be achieved if

$$\bar{y} = \frac{y_+ + y_-}{2} \qquad (10a)$$

and

$$S_y = \left| \frac{y_+ - y_-}{2} \right|. \qquad (10b)$$

This can be heuristically proven by noting that $f(x)$ can be expanded in Taylor's series (Harr, 1977) and the mean and variance calculated by

$$\bar{y} \approx f(\bar{x}) + \tfrac{1}{2} f''(\bar{x}) S_x^2 \qquad (11a)$$

$$S_y^2 \equiv [f'(\bar{x})]^2 S_x^2 \qquad (11b)$$

The validity of eqns. (10*a*) and (10*b*) can be demonstrated by assuming any arbitrary differentiable function $f(x)$, using eqn. 9 in eqns. (10*a*) and (10*b*), and showing the result equal to eqns. (11*a*) and (11*b*).

The simulated frost heave, y', is a function of many parameters, p_i including the input conditions x', i.e.

$$y' = f(p_1, p_2, \ldots, p_m, x'), \qquad (12)$$

where for the moment it is assumed the p_i are uncorrelated. Similar to the heuristic proof above for one variate, Rosenblueth deduced the general relationship

$$E[(y')^N] = \frac{1}{2^m}\,[(y'_{++\ldots m})^N + (y'_{-++\ldots m})^N + \ldots + (y'_{--\ldots m})^N] \qquad (13)$$

where there are m parameters to be considered, and N is the exponent (moment) of y'. The notation $y'_{++\ldots m}$ indicates the use of all sign permutations of

$$y' = f(\bar{p}_1 \pm S_{p_1}, \bar{p}_2 \pm S_{p_2}, \ldots, \bar{p}_m \pm S_{p_m}) \qquad (14)$$

where $\bar{p}_i$ is the mean of the ith parameter and S_{p_i} is the standard deviation of the parameter. The subscript sign is determined by the sign of S_p. The mean and variance of y' are computed in the usual fashion

$$\bar{y}' = E(y') \qquad (15)$$

$$V_{y'} = E[(y')^2] - [E(\bar{y}')]^2. \qquad (16)$$

Now suppose some or all of the p_i are correlated, Rosenblueth's method can be extended using the covariance (cov) statistic [Harr (1977)] as follows:

$$\rho_{g,h} = \rho_{h,g} = \frac{\mathrm{cov}(p_g, p_h)}{S_{p_g} S_{p_h}} \qquad (17)$$

where ρ is the covariance measure and the subscripts denote there are m random variables (parameters) that are correlated a pair at a time. Define a q-function such that there will be M of these functions given by

$$q_{ij\ldots m} = 1 + \sum_{\substack{g=1 \\ h=1}}^{M} \frac{|gh|}{gh}\,\delta_{g,h}\,\rho_{g,h} \qquad (18)$$

$$\delta_{g,h} = \begin{cases} 0, & |g| \geq |h| \\ 1, & |g| < |h| \end{cases}$$

where the $i,j \ldots m$ are all the permutations of the signs of the standard deviation of each parameter where each sign is attached to the subscript. The moments of y' are defined as

$$E[(y')^N] = \frac{1}{2^m}\sum_{0}^{M} (q_{ij\ldots m})(y_{ij\ldots m})^N \qquad (19)$$

and the first and second moments are computed as in eqns. (15) and (16). Equation (19) reduces to eqn. (13) in the event all ρ are zero (i.e. the p_i are all uncorrelated and $q = 1$).

Rosenblueth's method is a powerful tool that is ideally suited to the type of problem being considered. No prior assumptions are required concerning the probability distribution of the parameter variables. Only an estimate of parameter mean and coefficient of variation are necessary. This method requires only that the functional relationship between y' and x' need be specified, i.e. the deterministic model. The method is general, however, and is applicable to any deterministic model. Instead of the many costly simulations required by the commonly employed Monte Carlo method, only 2^m simulations are required using the present modification of Rosenblueth's method.

Given a measure of parameter variability the first and second moments of predicted frost heave can be readily computed, giving useful statistical information to establish a range of possible frost heave rather than a single deterministic value. While higher moments can be obtained to give some indication of skewness and peakedness, this information is not of great significance to the present objective.

The capability of extending knowledge by supposing we know nothing about the distribution of frost heave follows from Chebeshev's inequality

$$P[\bar{y} - hS_y \leqslant y \leqslant \bar{y} + hS_y] \geqslant 1 - \frac{1}{h^2} \tag{20}$$

For example, if two standard deviations are used (h=2), the probability that y is bounded by $\pm\, hS_y$ is greater than or equal to 75%. Now if it is assumed y is symmetrically distributed, Gauss' inequality may be applied

$$P[\bar{y} - hS_y \leqslant y \leqslant \bar{y} + hS_y] \geqslant 1 - \frac{4}{9h^2} \tag{21}$$

which says for h=2 there is a greater or equal probability of 89% that y is so bounded. Finally, given additional information the distribution of y can be further narrowed. A versatile distribution to assume is the beta distribution (Harr, 1977):

$$f(y) = \frac{\alpha!\beta!(b-a)^{\alpha+\beta+1}}{(\alpha+\beta+1)} (y-a)^{\alpha}(b-y)^{\beta} \tag{22}$$

where to find the α and β parameters one needs to know y, S_y, and a and b, the lower and upper bounds of the distribution. The parameters y and S_y are generated by Rosenblueth's method. The a and b parameters may be estimated by data such as Chamberlain (1980) has developed. He found that $a = \bar{y} - 2S_y$ and $b = \bar{y} + 3S_y$ for one field study. Once a beta distribution is determined, confidence limits and other desired statistical properties of $f(y)$ can be established (Harr, 1977).

Questions yet to be resolved include the question of stationarity, or in other words, how will the statistical properties of $f(y)$ vary with time? The second question concerns the nature of $f(y)$ for various soils. Can a single beta-distribution be found that is applicable to a class of soils, such as the so-called "frost-susceptible soils"?

APPLICATION OF MODEL

The model was applied to a set of laboratory data obtained from a vertical soil column equipped with temperature and water level control and instrumentation to measure soil temperatures and pore water pressures (described by Berg et al., 1980; Ingersoll and Berg, 1981). Two different uniform soils were placed in the column to obtain comparison data: a well known frost susceptible soil, Fairbanks silt, and a weakly frost-susceptible dirty gravel, West Lebanon gravel. Guymon et al. (1981) reported on comparisons of simulated frost heave and measured laboratory frost heave for both restrained and unrestrained cases using Fairbanks silt among other soils.

The simulation procedure consists of determining hydraulic parameters using remolded samples of the same soil employed in the frost heave column. Porosity and density are determined using standard techniques. A modified Tempe cell, as described by Ingersoll and Berg (1981), was used to determine the soil characteristic drying curve and the unsaturated hydraulic conductivity relationship. An average characteristic curve of two tests on Fairbanks silt is shown in Fig. 4 together with the parameters used to describe this curve in the model where

$$\theta_u = \frac{\theta_0}{A_w|\psi|^n + 1} \,. \tag{23}$$

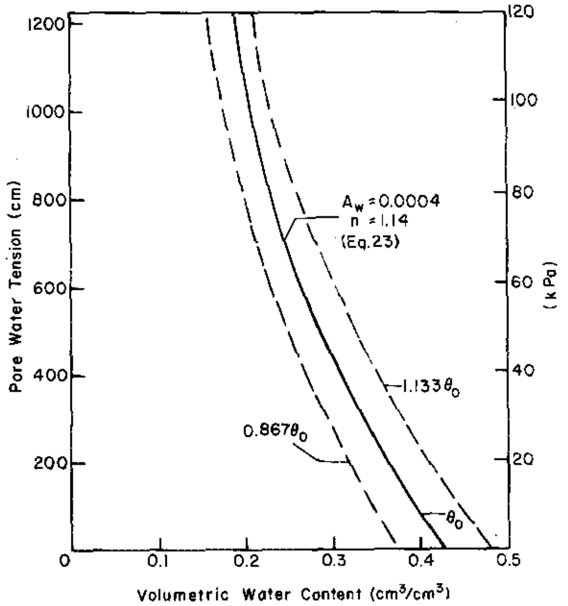

Fig. 4. Soil–water characteristic for Fairbanks silt (average of two tests).

These parameters fit the actual curve with less than a ±0.03 error. Unfrozen hydraulic conductivity versus pore water pressure, $K(\psi)$, for a sample of Fairbanks silt during a drying cycle, is shown in Fig. 5. The

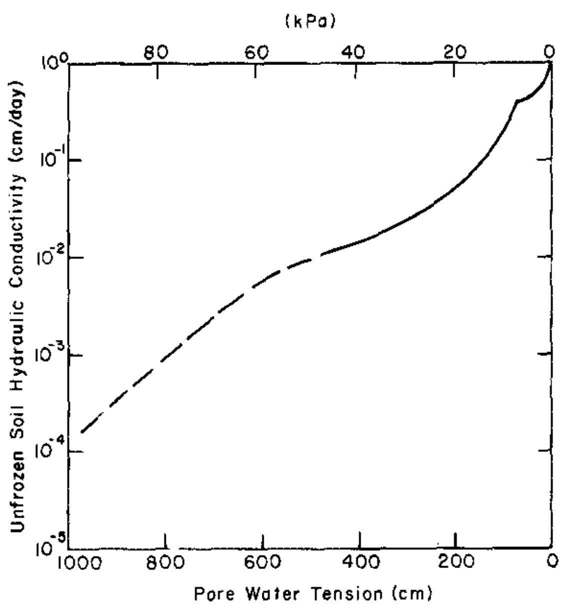

Fig. 5. Unfrozen hydraulic conductivity for Fairbanks silt (solid line derived for modified tempe cell, broken line derived from volumetric pressure plate extractor).

solid line fits data obtained from the modified Tempe cell and the broken line fits data obtained from a volumetric pressure plate extractor. The deterministic model uses unfrozen hydraulic conductivity data as a discrete table where intermediate values are determined by linear interpolation. The partly frozen soil hydraulic conductivity correction factor, E, where

$$K(\psi,\theta_i) = K(\psi)\cdot 10^{-E\theta_i} \tag{24}$$

was determined by calibration such that a deterministic simulation of frost heave closely approximated measured frost heave in the laboratory freezing column. Other parameters in Table 1 such as the thermal parameters, were assumed. Table 2 lists the parameters used in the "best" deterministic simulation of measured frost heave for both Fairbanks silt and West Lebanon gravel. It is assumed that these parameters represent the mean parameter values for the soil in the laboratory freezing column.

Boundary and initial conditions applied to the laboratory column were approximated as closely as possible in the deterministic model simulations. Because there is inherently some uncertainty concerning boundary conditions, a precise simulation was not expected or attempted.

TABLE 2

Mean soil parameters used in model simulations

Parameter	Soil	
	Fairbanks silt	W. Lebanon gravel
A_w	0.0004 [a,f]	0.119 [a]
n	1.14 [a, f]	0.50 [a]
$K(\psi=0)$	0.04 cm h^{-1} [a,d]	0.42 cm h^{-1} [a,e]
E	8 [b]	20 [b]
θ_n	0.150 [a]	0.082 [b]
θ_0	0.425 [a]	0.27 [a]
ρ_s	1.6 [a]	1.99 [a]
T_f	0°C [c]	0°C [c]
K_s	17 cal cm^{-1} °C^{-1} h^{-1} [c]	17 cal cm^{-1} °C^{-1} h^{-1} [c]
C_s	0.3 cal cm^{-3} °C^{-1} [c]	0.3 cal cm^{-3} °C^{-1} [c]

[a] Measured in laboratory.
[b] Determined by model calibration.
[c] Assumed.
[d] See Fig. 5 for complete relationship.
[e] Complete relationship not shown herein.
[f] Average of two laboratory tests.

TABLE 3

Comparison of measured unrestrained and simulated frost heave varying soil parameters for Fairbanks silt (cm)

Day	Measured	Best simulation	$T_f = -0.005$	$0.82\,K_s$	$1.33\,C_s$	$1.3\,K(\psi)$	$1.15\,\theta_n$	$1.13\,\theta_0$	$1.1\,E$
5	1.6	1.5	1.5	1.5	1.5	1.6	1.6	1.5	1.5
10	2.8	2.6	2.6	2.6	2.7	2.8	2.8	2.7	2.6
15	3.9	3.5	3.5	3.6	3.6	3.7	3.7	3.7	3.5
20	4.4	4.4	4.4	4.4	4.4	4.6	4.6	4.6	4.3
25	5.0	5.1	5.1	5.2	5.2	5.3	5.3	5.4	5.0
30	5.6	5.7	5.8	5.9	5.8	6.0	6.0	6.1	5.7

Table 3 compares simulated and measured unrestrained cumulative frost heave for the Fairbanks silt case. To evaluate the effect of a single-parameter variation while holding all other parameters at their assumed mean value, seven additional simulations are shown in Table 3. Although a substantial variation in K_s shows some sensitivity, it was assumed that thermal parameters would have a minor effect on frost heave simulation results for Fairbanks silt under the conditions of the laboratory tests. The reason for this is that phase change processes overshadow sensible heat processes in a freezing soil. Although not shown, parameter variations for Fairbanks silt have an insignificant effect on simulated frost penetration which very closely approximated measured frost penetration. Frost heave showed marked sensitivity to hydraulic parameter variations. Consequently, these parameters were selected for a more detailed analysis using Rosenblueth's method. The most sensitive parameters are porosity, unfrozen water content factor, and unfrozen hydraulic conductivity.

Varying porosity within reasonable bounds affects the results by modifying the mass accumulation term in the moisture transport equation (3) by modifying the available pore space in the soil for ice to develop without ice segregation occurring, and by modifying the soil–water characteristics (eqn. (23)). Varying porosity can also account for hysteresis (Fig. 4) which is not considered in the deterministic model. To include this phenomenon would necessitate the incorporation of memory into the model, significantly increasing the computation effort and probably not markedly improving the computation precision or certainty. Figure 4 shows a plus and minus 13.3% variation effect on the characteristic curve. Numerous measurements we have made on silts and similar soils suggest that hysteresis effects would be adequately bounded by such an approach. We believe there is now no reason to explicity include hysteresis in any soil phenomena related model. This process can be adequately dealt with by incorporation of a probabilistic model.

The so-called "unfrozen water content factor" controls the available space for pore ice to develop before ice segregation occurs. In the deterministic model, this parameter also determines the unrestrained pore water pressure at the bottom of the frozen zone (Fig. 2), thereby determining the hydraulic gradient and the rate water is drawn into the freezing zone. The balance between the rate of heat extraction and water importation to this zone is the controlling factor in the ice segregation processes as the deterministic model is conceived. For this reason, the hydraulic conductivity of the soil system is obviously an important, if not the most significant, parameter, although it is difficult to measure accurately for unsaturated fine-grained soils and is subject to considerable uncertainty. Very little work has been done on measuring hydraulic conductivity for partly frozen soils in the range of temperatures found in field soils under winter conditions.

The "best" simulation results for measured frost heave of West Lebanon gravel are shown in Figs. 6 and 7. The laboratory results for a slightly restrained soil (i.e. 0.5 lbf/in^2 or 3.45 kPa surcharge) were used for parameter calibration (see Table 2); the results are shown in Fig. 6. A restrained (i.e. 5.0 lbf/in^2 or 34.5 kPa surcharge) laboratory case with identical boundary conditions to the slightly restrained case was

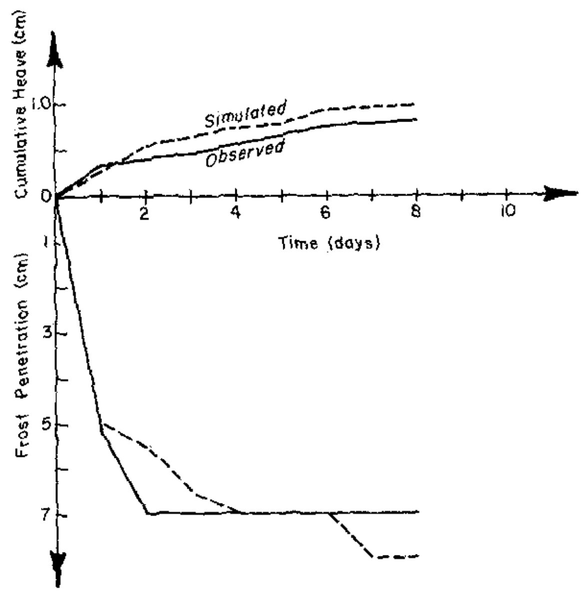

Fig. 6. Observed and simulated frost heave and frost penetration for West Lebanon gravel with a 0.5 lbf/in² (3.45 kPa) surcharge.

simulated without adjusting parameters. As can be seen from Fig. 7, the results are satisfactory.

The effect of soil density variations was not studied because this parameter has a minor effect on overburden pressures for shallow freezing cases with

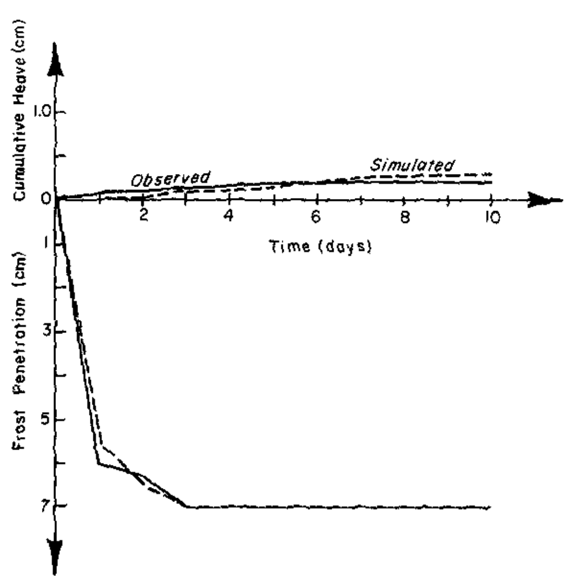

Fig. 7. Observed and simulated frost heave and frost penetration for West Lebanon gravel with a 5.0 lbf/in² (34.5 kPa) surcharge.

which we are concerned. Obviously density variations are highly important since porosity and the hydraulic parameters are closely correlated to density.

Freeze (1975) reviews some references that deal with porosity and hydraulic conductivity variations and Harr (1977) reviews some of the available literature on these and other soil parameter statistics. Warrick and Nielson (1980) review their own data as well as those of others for soil density, water content, particle size, and hydraulic conductivity variations. Schultze (1972) obtained a coefficient of variation for porosity of silt of 13.3% while Nielsen et al. (1973) obtained an average coefficient of variation of 10.0% for a clay–loam soil. They also obtained a coefficient of variation for the same soil for soil–water characteristics that ranged from 10% at low tensions to about 24% at moderate tensions (i.e. 200 cm of water). We know of no similar data for the unfrozen water content factor; however, we assume a similar range of behavior as for porosity. A coefficient of variation of 15% is probably adequate to describe most frost susceptible soils. Nielsen et al. (1973) performed extensive analysis on the variability of unsaturated hydraulic conductivity for clay–loam soil. They report a coefficient of variation that ranges from 100 to 450% for field variations. Laboratory measured variations for the same soil might be on the lower end of this range (i.e. 100%). Very little work has been done on determining partially frozen soil hydraulic conductivity, let alone determining measurement or sampling errors and field variability. It is probably safe to assume considerably more variability for frozen soil than for an unfrozen soil. For lack of more definitive data, we assume a coefficient of variation of 500% for frozen soil hydraulic conductivity. If the E-factor varies by 10%, about a 500% coefficient of variation in frozen soil hydraulic conductivity is obtained when we couple the E-factor variation to the unfrozen hydraulic conductivity variation.

Although there may be some autocorrelation between hydraulic parameters, the correlation coefficient was taken to be zero (i.e. $\rho = 0$, eqn. 17)). Limited attempts to correlate parameters showed a very weak correlation between porosity and hydraulic conductivity; however, sufficient data were not available to draw definitive conclusions.

Equations (13)–(16) were used as previously described to generate the mean and variance of simulated heave due to parameter variation. Four parameters were used in the procedure where $m = 4$ in eqn. (13) and

$$\left.\begin{aligned} \theta_0 &= (1 \pm CV)\,\overline{\theta}_0 \\ \theta_n &= (1 \pm CV)\,\overline{\theta}_n \\ K(\psi) &= (1 \pm CV)\,\overline{K}(\psi) \\ E &= (1 \pm CV)\,\overline{E} \end{aligned}\right\} \tag{25}$$

where the bar denotes the assumed mean value used in the "best" calibration simulation. All other parameters were held constant.

Boundary conditions measured in the laboratory were closely approximated in the simulations. Boundary condition errors were effectively eliminated by using the same boundary and initial conditions for each separate simulation. However, because there are boundary condition errors we can only comment on variations around a mean and cannot comment on simulation error (eqn. (1)), resulting from parameter variability.

Table 4 presents results for a 30-day simulation assuming parameter coefficients of variation for θ_0, θ_n, E, and $K(\psi)$ of 13.3, 15, 10, and 30%, respectively, and using parameters for unrestrained Fairbanks silt. As can be seen the mean frost heave using Rosenblueth's method closely approximates the "best" simulation shown in Table 3. The coefficient of variation of simulated frost heave derived from employing Rosenblueth's method is essentially stationary after some possible initial numerical instability. Although we have not attempted longer simulations using Rosenblueth's method, it is reasonable to expect a stationary coefficient of variation. Based upon field data obtained by Chamberlain (1980), we have assumed a beta-distribution lower bound of three standard deviations (Chamberlain actually obtained two) and an upper bound of four standard deviations. The α and β parameters of the beta-distribution are derived after Harr (1977) and are listed for each simulation day tabulated.

As can be seen, α approximately equals a constant 3.2, and β about equals a constant 4.8 in the last two-thirds of the simulation. The probability confidence limits are also shown in Table 4 for two standard deviations. The results are slightly skewed since we deliberately choose a slightly skewed distribution (Harr, 1977). Other confidence bounds can be easily obtained by integrating eqn. (22) between any desired limits.

Additional simulations were performed for unrestrained Fairbanks silt and slightly restrained and restrained West Lebanon gravel. A summary of the normalized results is shown in Table 5. The results shown in Table 4 are normalized and repeated in Table 5 on the first line. Similar to the results shown in Table 4, the coefficient of variation of simulated frost heave becomes essentially stationary after the first five days. As can be seen from Table 5, there is considerable variation in the coefficient of variation depending on the soil simulated, the magnitude of parameter variations, and the surcharge condition. Nevertheless, the α and β parameters of the beta-

TABLE 4

Simulated frost heave for unrestrained Fairbanks silt using Rosenblueth's method and varying $\theta_0 \pm 13.3\%$, $\theta_n \pm 15\%$, $E \pm 10\%$, and $K(\psi) \pm 30\%$

Day	Mean cumulative frost heave (cm)	Standard deviation	Coefficient of variation (%)	$a=\bar{y}-3S_y$	$b=\bar{y}+4S_y$	α	β	$P[y\leqslant\bar{y}+2S_y]$	$P[y\geqslant\bar{y}-2S_y]$
5	1.5	0.16	11	1.0	2.1	3.55	4.47	0.96	0.97
10	2.6	0.24	9	1.9	3.6	3.85	5.93	0.97	0.99
15	3.5	0.31	9	2.6	4.7	3.24	4.65	0.96	0.98
20	4.3	0.38	9	3.2	5.8	3.24	4.78	0.97	0.99
25	5.1	0.44	9	3.8	6.9	3.24	4.87	0.96	0.98
30	5.8	0.51	9	4.3	7.8	3.29	4.72	0.96	0.98

TABLE 5

Simulated frost heave statistics using Rosenblueth's method and an assumed beta-distribution for unrestrained Fairbanks silt and unrestrained and restrained West Lebanon gravel

Soil	Parameter coefficient of variation				Normalized simulated frost heave			a/y	b/y	α	β	$P\{\bar{y}-2S_y \leq y \leq \bar{y}+2S_y\}$
	θ_0	θ_n	E	$K(\psi)$	CV (%)	Min	Max					
Fairbanks silt	13.3	15	10	30	9	0.86	1.12	0.74	1.34	3.3	4.8	96
Fairbanks silt	20	20	20	50	17	0.76	1.24	0.48	1.66	3.6	5.5	97
Fairbanks silt	13.3	15	10	100	95	0.05	2.10	−1.84 [a]	4.79	3.7	5.3	97
W. Lebanon gravel – 0.5 psi surcharge	13.3	15	10	30	20	0.67	1.33	0.39	1.81	3.7	5.3	97
W. Lebanon gravel – 5.0 psi surcharge	13.3	15	10	30	107	0	2.91	−2.21 [a]	5.28	3.4	5.2	97
W. Lebanon gravel – 0.5 psi surcharge	13.3	15	10	100	103	0	2.45	−2.09 [a]	5.12	3.4	4.9	97

[a] A minus value used to derive beta distribution parameters.

distribution are quite similar, suggesting a universal frequency distribution is applicable to the model we are employing. An α of about 3.5 and a β of about 5.0 will generally reproduce the same coefficient of variation tabulated, using the correct a and b limits, with only a minor difference in results. The last column of Table 5 lists the smallest percent probability that the computed heave will lie within two standard deviations of the mean. Recall from Table 4 that confidence limits are slightly skewed. To derive a meaningful beta-distribution it was necessary to assume a lower limit a that was negative in several cases. Obviously, the lowest possible value of frost heave must be zero.

TABLE 6

Comparison of simulated heave for non-uniform and uniform soil profiles with average parameters derived from non-uniform profile

Day	Cumulative frost heave (cm)	
	Uniform profile	*Non-uniform profile*
5	2.0	2.1
10	3.1	3.2
15	3.9	3.9
20	4.4	4.4
25	4.8	4.8
30	5.0	5.0

Although the results are not presented here, parameter variations were found to have little direct effect upon simulated frost penetration. The reason for this is that latent heat effects dominate the thermal process and we assume that the thermal coefficient of phase change is a constant and equal to the value for bulk water. The computation of frost penetration is influenced by the amount of frost heave estimated. Thus, an error in computed frost heave will influence the estimate of frost penetration.

A non-uniform soil profile situation was examined to demonstrate the feasibility of modeling a layered soil profile as an averaged uniform profile. Because we have not conducted laboratory studies of non-uniform, layered, soil profiles, we assumed a situation similar to the case represented in Table 4. Slightly different boundary conditions were used since the computer code for the layered case uses a somewhat different boundary condition simulator. First it was assumed that the soil profile from surface down was a 5 cm layer of sandy soil, a 5 cm layer of silty soil, a 5 cm layer of clayey silt soil, and finally a 30 cm layer of silty soil. Representative hydraulic parameters were applied and frost heave simulated for 30 days of real time. The resulting heave was compared to a similar simulation using exactly the same boundary conditions but assuming a uniform soil profile with hydraulic parameters about equal to the average of those used in the layer simulation. The results of both simulations are shown in Table 6. In view of the often used approximations to represent a prototype soil profile (i.e. simplify a layered soil by assuming a homogeneous uniform soil profile), the results of Table 6 can be viewed with some optimism. The simulated frost depth at the end of the simulation was more than 17 cm below the original ground surface so that freezing had completely penetrated through the first three layers of the soil profile. Both results are almost identical and the non-uniform variation is certainly well within the confidence limits shown for the uniform soil parameter variability studies.

DISCUSSION

A powerful general tool is offered to evaluate the effects of parameter variability upon deterministic computations of any process in which parameters have a known variation. The method was applied to the frost heave simulation or analysis problem.

It was shown that thermal parameter variations had a less important effect upon simulated frost heave than hydraulic parameters. The reason for this is that thermal processes are dominated by the phase change process.

Table 5 indicates that the most important parameter variation is hydraulic conductivity. For gravels, the coefficient of variation of frost heave will about equal the coefficient of variation of unfrozen hydraulic conductivity. For silts, the coefficient of variation for simulated frost heave will be less than half of that for gravels at smaller variations of unsaturated hydraulic conductivity and is about the same as for gravels at larger variations of unsaturated hydraulic conductivity. The results obtained for West Lebanon gravel with a 5.0 lbf/in^2 (34.5 kPa) surcharge should be viewed as a special case since West Lebanon gravel is marginally frost-susceptible and a 5.0 lbf/in^2 surcharge is about the critical pressure to restrain frost heave completely. In this situation frost heave is much more sensitive to parameter variations than when the system is only moderately restrained.

It appears that a conservative universal approach to defining the coefficient of variation of simulated deterministic heave would be to use an assumed coefficient of variation for hydraulic conductivity, use an assumed number of standard deviations of variation to compute the lower beta-distribution bound, use the definition of mean and variance of the beta-distribution as defined by Harr (1977), and compute the coefficient of variation of frost heave assuming $\alpha = 3.5$ and $\beta = 5.0$. One could safely be assured that computed heave would lie within two standard deviations with a 95% probability. Such a computation would not apply to a critically restrained soil.

Comparing the simulation results to Chamberlain's (1980) data and the Albany County Airport data we have collected shows some striking similarities. If one assumes −3 and +4 standard deviations for the beta distribution minimum and maximum, as was assumed for simulated heave, similar α and β parameters for the beta-distribution are obtained. For Chamberlain's data, $\alpha = 3.6$ and $\beta = 5.2$ were obtained and for the Albany County Airport data, $\alpha = 3.8$ and $\beta = 5.4$ were obtained. This similarity to the simulated frost heave results suggests a strong justification for our results and our contention that deterministic solutions alone are not adequate. A probabilistic model coupled with a deterministic model is required. Furthermore, the results strongly suggest that the deterministic approach advocated is valid.

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